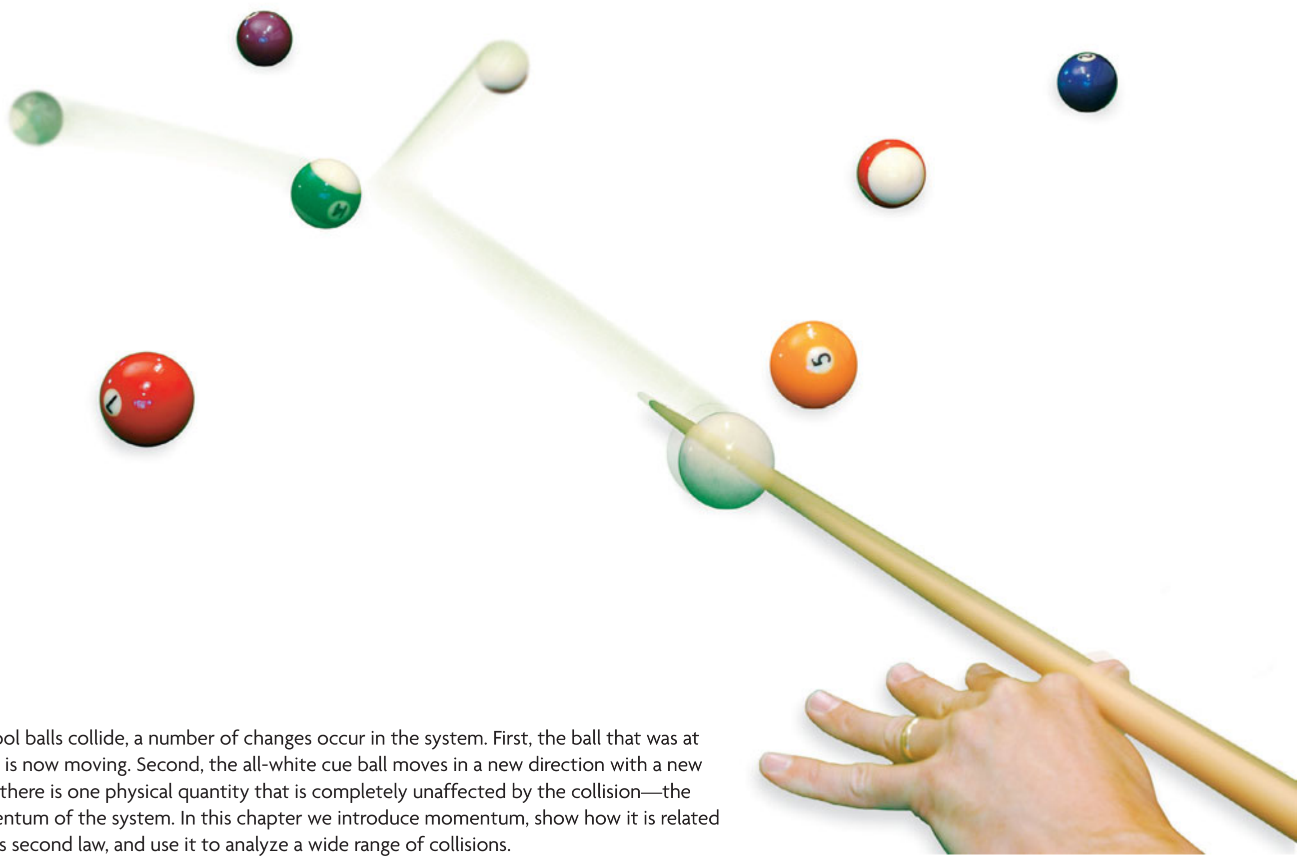
9 Linear Momentum and Collisions



As these pool balls collide, a number of changes occur in the system. First, the ball that was at rest initially is now moving. Second, the all-white cue ball moves in a new direction with a new speed. Still there is one physical quantity that is completely unaffected by the collision—the total momentum of the system. In this chapter we introduce momentum, show how it is related to Newton's second law, and use it to analyze a wide range of collisions.

Conservation laws play a central role in physics. In this chapter we introduce the concept of *momentum* and show that it, like energy, is a conserved quantity. Nothing we can do—in fact, nothing that can occur in nature—can change the total energy or the total momentum of the universe.

As with conservation of energy, we shall see that the conservation of momentum provides a powerful way of approaching a variety of problems that would be extremely difficult to solve using Newton's laws directly. In particular, problems involving the collision of two or more objects—such as a baseball bat striking a ball or one car bumping into another at an intersection—are especially well suited to a momentum approach. Finally, we introduce the concept of the *center of mass* and show that it allows us to extend many of the results that have been obtained for point particles to systems involving more realistic objects.

9–1 Linear Momentum

Imagine for a moment that you are sitting at rest on a skateboard that can roll without friction on a smooth surface. If you catch a heavy, slow-moving ball tossed to you by a friend, you begin to move. If, on the other hand, your friend tosses you a light, yet fast-moving ball, the net effect may be the same—that is, catching a lightweight ball moving fast enough will cause you to move with the same speed as when you caught the heavy ball.

In physics, the previous observations are made precise by defining a quantity called the **linear momentum**, $\vec{\mathbf{p}}$, which is defined as the product of the mass m and velocity $\vec{\mathbf{v}}$ of an object:

Definition of Linear Momentum, $\vec{\mathbf{p}}$

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \tag{9–1}$$

SI unit: kg · m/s

In our example, if the heavy ball has twice the mass of the light ball but the light ball has twice the speed of the heavy ball, the momenta of the two balls are equal in magnitude. We can see from Equation 9–1 that the units of linear momentum are simply the units of mass times the units of velocity: kg · m/s. There is no special shorthand name given to this combination of units.

It is important to note that a constant *linear* momentum $\vec{\mathbf{p}}$ is the momentum of an object of mass m that is *moving in a straight line* with a velocity $\vec{\mathbf{v}}$. In Chapter 11 we introduce a similar quantity to describe the momentum of an object that rotates. This momentum will be referred to as the *angular momentum*. In general, when we simply say momentum, we are referring to the linear momentum $\vec{\mathbf{p}}$. We will always specify angular momentum when referring to the momentum associated with rotation.

Because the velocity $\vec{\mathbf{v}}$ is a vector with both a magnitude and a direction, so too is the momentum, $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. The next Exercise gives some feeling for the *magnitude* of the momentum, $p = mv$, for everyday objects.

EXERCISE 9–1

(a) A 1180-kg car drives along a city street at 30.0 miles per hour (13.4 m/s). What is the magnitude of the car's momentum? **(b)** A major-league pitcher can give a 0.142-kg baseball a speed of 101 mi/h (45.1 m/s). Find the magnitude of the baseball's momentum.

SOLUTION

a. Using $p = mv$, we find

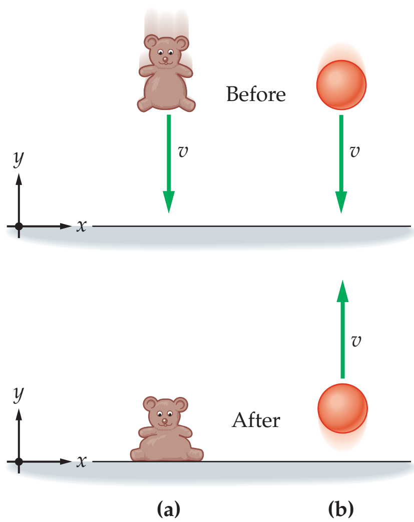
$$p_c = m_c v_c = (1180\ \text{kg})(13.4\ \text{m/s}) = 15{,}800\ \text{kg} \cdot \text{m/s}$$

b. Similarly,

$$p_b = m_b v_b = (0.142\ \text{kg})(45.1\ \text{m/s}) = 6.40\ \text{kg} \cdot \text{m/s}$$

As an illustration of the vector nature of momentum, consider the situations shown in **Figures 9–1 (a)** and **(b)**. In Figure 9–1 (a), a 0.10-kg beanbag bear is dropped to the floor, where it hits with a speed of 4.0 m/s and sticks. In Figure 9–1 (b) a 0.10-kg rubber ball also hits the floor with a speed of 4.0 m/s, but in this case the ball bounces upward off the floor. Assuming an ideal rubber ball, its initial upward speed is 4.0 m/s. Now the question in each case is, "What is the change in momentum?"

To approach the problem systematically, we introduce a coordinate system as shown in Figure 9–1. With this choice, we can see that neither object has momentum in the x direction; thus we need only consider the y component of momentum, p_y. The problem, therefore, is one-dimensional; still, we must be careful about the sign of p_y.



▲ **FIGURE 9–1 Change in momentum**
A beanbag bear and a rubber ball, with the same mass m and the same downward speed v, hit the floor. **(a)** The beanbag bear comes to rest on hitting the floor. Its change in momentum is mv upward. **(b)** The rubber ball bounces upward with a speed v. Its change in momentum is $2mv$ upward.

We begin with the beanbag. Just before hitting the floor, it moves downward (that is, in the negative y direction) with a speed of $v = 4.0\ \mathrm{m/s}$. Letting m stand for the mass of the beanbag, we find that the initial momentum is

$$p_{y,\mathrm{i}} = m(-v)$$

After landing on the floor, the beanbag is at rest; hence, its final momentum is zero:

$$p_{y,\mathrm{f}} = m(0) = 0$$

Therefore the change in momentum is

$$\Delta p_y = p_{y,\mathrm{f}} - p_{y,\mathrm{i}} = 0 - m(-v) = mv$$
$$= (0.10\ \mathrm{kg})(4.0\ \mathrm{m/s}) = 0.40\ \mathrm{kg \cdot m/s}$$

Note that the change in momentum is positive—that is, in the upward direction. This makes sense because, before the bag landed, it had a negative (downward) momentum in the y direction. In order to increase the momentum from a negative value to zero, it is necessary to add a positive (upward) momentum.

Next, consider the rubber ball in Figure 9–1 (b). Before bouncing, its momentum is

$$p_{y,\mathrm{i}} = m(-v)$$

the same as for the beanbag. After bouncing, when the ball is moving in the upward (positive) direction, its momentum is

$$p_{y,\mathrm{f}} = mv$$

As a result, the change in momentum for the rubber ball is

$$\Delta p_y = p_{y,\mathrm{f}} - p_{y,\mathrm{i}} = mv - m(-v) = 2mv$$
$$= 2(0.10\ \mathrm{kg})(4.0\ \mathrm{m/s}) = 0.80\ \mathrm{kg \cdot m/s}$$

PROBLEM-SOLVING NOTE

Coordinate Systems

Be sure to draw a coordinate system for momentum problems, even if the problem is only one-dimensional. It is important to use the coordinate system to assign the correct sign to velocities and momenta in the system.

This is *twice* the change in momentum of the beanbag! The reason is that in this case, the momentum in the y direction must first be increased from $-mv$ to 0, then increased again from 0 to mv. For the beanbag, the change was merely from $-mv$ to 0.

Note how important it is to be careful about the vector nature of the momentum and to use the correct sign for p_y. Otherwise, we might have concluded—erroneously—that the rubber ball had zero change in momentum, since the *magnitude* of its momentum was unchanged by the bounce. In fact, its momentum does change due to the change in its *direction* of motion.

One additional point: Since momentum is a vector, the total momentum of a system of objects is the *vector* sum of the momenta of all the objects. That is,

$$\vec{\mathbf{p}}_{\mathrm{total}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3 + \cdots \qquad 9\text{–}2$$

This is illustrated for the case of three objects in the following Example.

EXAMPLE 9–1 DUCK, DUCK, GOOSE: ADDING MOMENTA

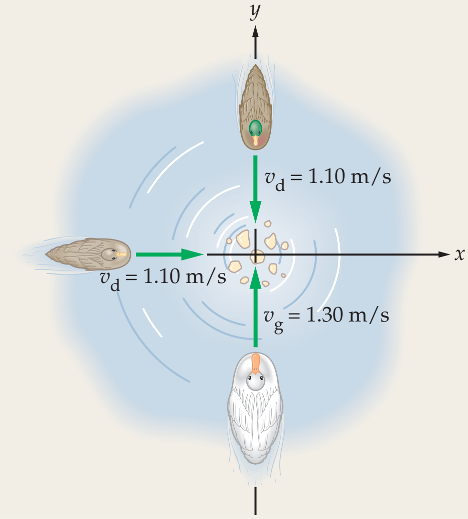
At a city park, a person throws some bread into a duck pond. Two 4.00-kg ducks and a 9.00-kg goose paddle rapidly toward the bread, as shown in our sketch. If the ducks swim at 1.10 m/s, and the goose swims with a speed of 1.30 m/s, find the magnitude and direction of the total momentum of the three birds.

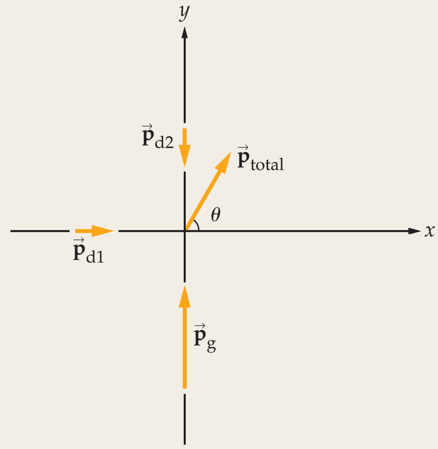
PICTURE THE PROBLEM

In our sketch we place the origin where the bread floats on the water. Note that duck 1 swims in the positive x direction, duck 2 swims in the negative y direction, and the goose swims in the positive y direction. Therefore, $\vec{\mathbf{p}}_{\mathrm{d}1} = m_\mathrm{d} v_\mathrm{d} \hat{\mathbf{x}}$, $\vec{\mathbf{p}}_{\mathrm{d}2} = -m_\mathrm{d} v_\mathrm{d} \hat{\mathbf{y}}$, and $\vec{\mathbf{p}}_\mathrm{g} = m_\mathrm{g} v_\mathrm{g} \hat{\mathbf{y}}$, where $v_\mathrm{d} = 1.10\ \mathrm{m/s}$, $m_\mathrm{d} = 4.00\ \mathrm{kg}$, $v_\mathrm{g} = 1.30\ \mathrm{m/s}$, and $m_\mathrm{g} = 9.00\ \mathrm{kg}$. The total momentum, $\vec{\mathbf{p}}_{\mathrm{total}}$, points at an angle θ relative to the positive x axis.

STRATEGY

Write the momentum of each bird as a vector, using unit vectors in the x and y directions. Next, sum these vectors component by component to find the total momentum. Finally, use the components of the total momentum to calculate its magnitude and direction.





SOLUTION

1. Use x and y unit vectors to express the momentum of each bird in vector form:

$$\vec{\mathbf{p}}_{d1} = m_d v_d \hat{\mathbf{x}} = (4.00\text{ kg})(1.10\text{ m/s})\hat{\mathbf{x}} = (4.40\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}}$$

$$\vec{\mathbf{p}}_{d2} = -m_d v_d \hat{\mathbf{y}} = -(4.00\text{ kg})(1.10\text{ m/s})\hat{\mathbf{y}} = -(4.40\text{ kg}\cdot\text{m/s})\hat{\mathbf{y}}$$

$$\vec{\mathbf{p}}_{g} = m_g v_g \hat{\mathbf{y}} = (9.00\text{ kg})(1.30\text{ m/s})\hat{\mathbf{y}} = (11.7\text{ kg}\cdot\text{m/s})\hat{\mathbf{y}}$$

2. Sum the momentum vectors to obtain the total momentum:

$$\vec{\mathbf{p}}_{total} = \vec{\mathbf{p}}_{d1} + \vec{\mathbf{p}}_{d2} + \vec{\mathbf{p}}_{g} = (4.40\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}} + [-4.40\text{ kg}\cdot\text{m/s} + 11.7\text{ kg}\cdot\text{m/s}]\hat{\mathbf{y}} = (4.40\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}} + (7.30\text{ kg}\cdot\text{m/s})\hat{\mathbf{y}}$$

3. Calculate the magnitude of the total momentum:

$$p_{total} = \sqrt{p_{total,x}^2 + p_{total,y}^2} = \sqrt{(4.40\text{ kg}\cdot\text{m/s})^2 + (7.30\text{ kg}\cdot\text{m/s})^2} = 8.52\text{ kg}\cdot\text{m/s}$$

4. Calculate the direction of the total momentum:

$$\theta = \tan^{-1}\left(\frac{p_{total,y}}{p_{total,x}}\right) = \tan^{-1}\left(\frac{7.30\text{ kg}\cdot\text{m/s}}{4.40\text{ kg}\cdot\text{m/s}}\right) = 58.9°$$

INSIGHT

Note that the momentum of each bird depends only on its mass and velocity; it is independent of the bird's location. In addition, we observe that the magnitude of the total momentum is less than the sum of the magnitudes of each bird's momentum individually. This is generally the case when dealing with vector addition—the only exception is when all vectors point in the same direction.

PRACTICE PROBLEM

Should the speed of the goose be increased or decreased if the total momentum of the three birds is to point in the positive x direction? Verify your answer by calculating the required speed. [**Answer:** The goose's speed must be decreased. Setting the momentum of the goose equal to minus the momentum of duck 2 yields $v_g = 0.489$ m/s.]

Some related homework problems: Problem 1, Problem 2, Problem 3

9–2 Momentum and Newton's Second Law

In Chapter 5 we introduced Newton's second law:

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

As mentioned, this expression is valid only for objects that have constant mass. The more general law, which holds even if the mass changes, is expressed in terms of momentum. In fact, Newton's original statement of the second law was in just this form:

Newton's Second Law

$$\sum \vec{\mathbf{F}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t} \qquad 9\text{–}3$$

That is, the net force acting on an object is equal to the change in its momentum divided by the time interval during which the change occurs—in other words, the net force is the rate of change of momentum with time.

To show the connection between these two statements of the second law, consider the change in momentum, $\Delta \vec{\mathbf{p}}$. Since $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, we have

$$\Delta \vec{\mathbf{p}} = \vec{\mathbf{p}}_{\mathrm{f}} - \vec{\mathbf{p}}_{\mathrm{i}} = m_{\mathrm{f}}\vec{\mathbf{v}}_{\mathrm{f}} - m_{\mathrm{i}}\vec{\mathbf{v}}_{\mathrm{i}}$$

However, if the mass is constant, so that $m_{\mathrm{f}} = m_{\mathrm{i}} = m$, it follows that the change in momentum is simply m times $\Delta \vec{\mathbf{v}}$:

$$\Delta \vec{\mathbf{p}} = m_{\mathrm{f}}\vec{\mathbf{v}}_{\mathrm{f}} - m_{\mathrm{i}}\vec{\mathbf{v}}_{\mathrm{i}} = m(\vec{\mathbf{v}}_{\mathrm{f}} - \vec{\mathbf{v}}_{\mathrm{i}}) = m\Delta \vec{\mathbf{v}}$$

As a result, Newton's second law, for objects of constant mass, can be written as follows:

$$\sum \vec{\mathbf{F}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t} = m\frac{\Delta \vec{\mathbf{v}}}{\Delta t}$$

Finally, recall that acceleration is the rate of change of velocity with time:

$$\vec{\mathbf{a}} = \frac{\Delta \vec{\mathbf{v}}}{\Delta t}$$

Therefore, we can write Equation 9–3 as

$$\sum \vec{\mathbf{F}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t} = m\vec{\mathbf{a}} \qquad 9\text{–}4$$

Hence, the two statements are equivalent if the mass is constant.

It should be noted, however, that $\Sigma \vec{\mathbf{F}} = \Delta \vec{\mathbf{p}}/\Delta t$ is the general form of Newton's second law, and that it is valid no matter how the mass may vary. In the remainder of this chapter we use this form of the second law to investigate the connections between forces and changes in momentum.

9–3 Impulse

The pitcher delivers a fastball, the batter takes a swing, and with a crack of the bat the ball that was approaching home plate at 95.0 mi/h is now heading toward the pitcher at 115 mi/h. In the language of physics, we say that the bat has delivered an **impulse,** $\vec{\mathbf{I}}$, to the ball.

During the brief time the ball and bat are in contact—perhaps as little as a thousandth of a second—the force between them rises rapidly to a large value, as shown in **Figure 9–2**, then falls back to zero as the ball takes flight. It would be almost impossible, of course, to describe every detail of the way the force varies with time. Instead, we focus on the average force exerted by the bat, $\vec{\mathbf{F}}_{\mathrm{av}}$, which is also shown in Figure 9–2. The impulse, then, is defined to be $\vec{\mathbf{F}}_{\mathrm{av}}$ times the length of time, Δt, that the ball and bat are in contact, which is simply the area under the force-versus-time curve:

Definition of Impulse, $\vec{\mathbf{I}}$

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\mathrm{av}}\,\Delta t \qquad 9\text{–}5$$

SI unit: $\mathrm{N \cdot s = kg \cdot m/s}$

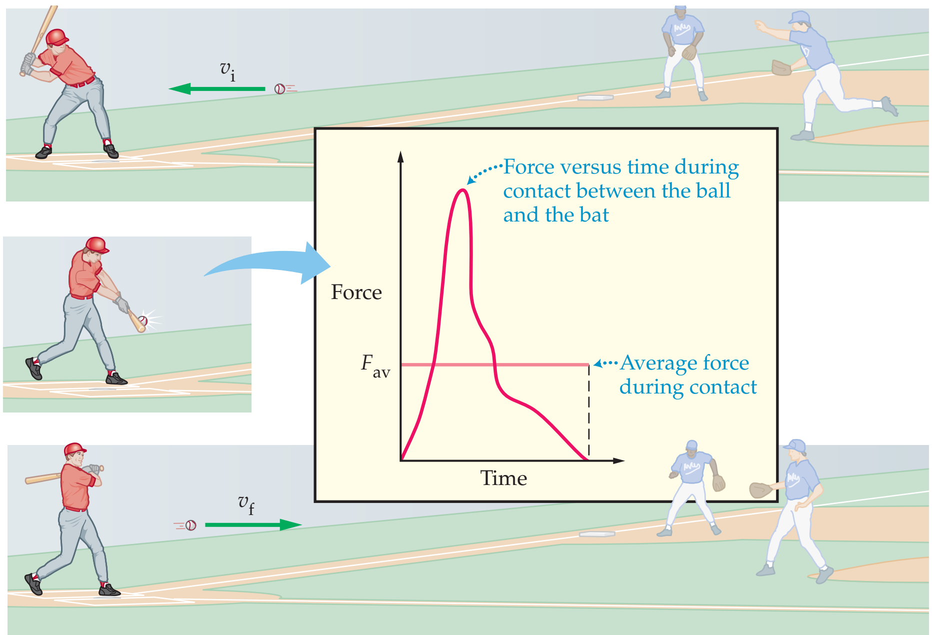


FIGURE 9–2 The average force during a collision

The force between two objects that collide, as when a bat hits a baseball, rises rapidly to very large values, then drops again to zero in a matter of milliseconds. Rather than try to describe the complex behavior of the force, we focus on its average value, F_{av}. Note that the area under the F_{av} rectangle is the same as the area under the actual force curve.

Note that impulse is a vector and that it points in the same direction as the average force. In addition, its units are $N \cdot s = (kg \cdot m/s^2) \cdot s = kg \cdot m/s$, the same as the units of momentum.

It is no accident that impulse and momentum have the same units. In fact, rearranging Newton's second law, Equation 9–3, we see that the average force times Δt is simply the change in momentum of the ball due to the bat:

$$\vec{\mathbf{F}}_{av} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t}$$

$$\vec{\mathbf{F}}_{av}\,\Delta t = \Delta \vec{\mathbf{p}}$$

Hence, in general, impulse is just the change in momentum:

Momentum–Impulse Theorem

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{av}\,\Delta t = \Delta \vec{\mathbf{p}} \qquad 9\text{–}6$$

For instance, if we know the impulse delivered to an object—that is, its change in momentum—and the time interval during which the change occurs, we can find the average force that caused the impulse.

As an example, let's calculate the impulse given to the baseball considered at the beginning of this section, as well as the average force between the ball and the bat. First, set up a coordinate system with the positive x axis pointing from home plate toward the pitcher's mound, as indicated in **Figure 9–3**. If the ball's mass is 0.145 kg, its initial momentum—which is in the negative x direction—is

$$\vec{\mathbf{p}}_i = -mv_i\hat{\mathbf{x}} = -(0.145\text{ kg})(95.0\text{ mi/h})\left(\frac{0.447\text{ m/s}}{1\text{ mi/h}}\right)\hat{\mathbf{x}} = -(6.16\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}}$$

Immediately after the hit, the ball's final momentum is in the positive x direction:

$$\vec{\mathbf{p}}_f = mv_f\hat{\mathbf{x}} = (0.145\text{ kg})(115\text{ mi/h})\left(\frac{0.447\text{ m/s}}{1\text{ mi/h}}\right)\hat{\mathbf{x}} = (7.45\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}}$$

The impulse, then, is

$$\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = [7.45\text{ kg}\cdot\text{m/s} - (-6.16\text{ kg}\cdot\text{m/s})]\hat{\mathbf{x}} = (13.6\text{ kg}\cdot\text{m/s})\hat{\mathbf{x}}$$

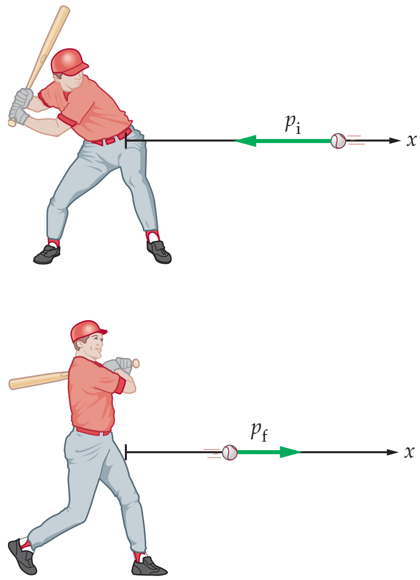


FIGURE 9–3 Hitting a baseball

A batter hits a ball, sending it back toward the pitcher's mound. The impulse delivered to the ball by the bat changes the ball's momentum from $-p_i\hat{\mathbf{x}}$ to $p_f\hat{\mathbf{x}}$.

REAL-WORLD PHYSICS

The force between a ball and a bat

▲ When a softball is hit by a bat (top), an enormous force (thousands of newtons) acts for a very short period of time—perhaps only a few ms. During this time, the ball is dramatically deformed by the impact. To keep the same thing from happening to a pole vaulter, who must fall nearly 20 feet after clearing the bar (bottom), a deeply padded landing area is provided. The change in the pole vaulter's momentum as he is brought to a stop, $mv = F\Delta t$, is the same whether he lands on a mat or on concrete. However, the padding is very yielding, greatly prolonging the time Δt during which he is in contact with the mat. The corresponding force on the vaulter is thus markedly decreased.

If the ball and bat are in contact for 1.20 ms $= 1.20 \times 10^{-3}$ s, a typical time, the average force is

$$\vec{\mathbf{F}}_{\text{av}} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t} = \frac{\vec{\mathbf{I}}}{\Delta t} = \frac{(13.6\ \text{kg}\cdot\text{m/s})\hat{\mathbf{x}}}{1.20 \times 10^{-3}\ \text{s}} = (1.13 \times 10^4\ \text{N})\hat{\mathbf{x}}$$

Note that the average force is in the positive x direction; that is, toward the pitcher, as expected. In addition, the magnitude of the average force is remarkably large. In everyday units, the force between the ball and the bat is more than 2500 pounds! This explains why the ball is observed in high-speed photographs to deform significantly during a hit—the force is so large that, for an instant, it partially flattens the ball. Finally, notice that the weight of the ball, which is only about 0.3 lb, is completely negligible compared to the forces involved during the hit.

In problems that are strictly one-dimensional, we can drop the vector notation when dealing with impulse. However, we must still be careful about the signs of the various quantities in the system. This is illustrated in the following Active Example.

ACTIVE EXAMPLE 9–1 FIND THE FINAL SPEED OF THE BALL

A 0.144-kg baseball is moving toward home plate with a speed of 43.0 m/s when it is bunted (hit softly). The bat exerts an average force of 6.50×10^3 N on the ball for 1.30 ms. The average force is directed toward the pitcher, which we take to be the positive x direction. What is the final speed of the ball?

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Relate change in momentum to impulse (Equation 9–5):	$\Delta p = p_{\text{f}} - p_{\text{i}} = I = F_{\text{av}}\,\Delta t$
2. Solve for the final momentum:	$p_{\text{f}} = F_{\text{av}}\,\Delta t + p_{\text{i}}$
3. Calculate the initial momentum:	$p_{\text{i}} = -6.19\ \text{kg}\cdot\text{m/s}$
4. Calculate the impulse:	$I = F_{\text{av}}\,\Delta t = 8.45\ \text{kg}\cdot\text{m/s}$
5. Use these results to find the final momentum:	$p_{\text{f}} = 2.26\ \text{kg}\cdot\text{m/s}$
6. Divide by the mass to find the final velocity:	$v_{\text{f}} = p_{\text{f}}/m = 15.7\ \text{m/s}$

INSIGHT

With our choice of coordinate system, we see that the initial momentum of the ball was in the negative x direction. The impulse applied to the ball, however, resulted in a final momentum (and velocity) in the positive x direction.

YOUR TURN

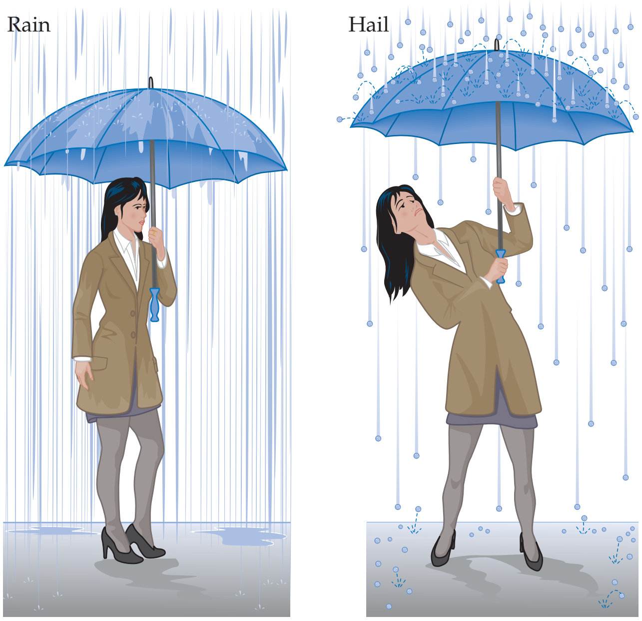
Suppose the bat is in contact with the ball for 2.60 ms rather than 1.30 ms. What is the final speed of the ball in this case?

(Answers to **Your Turn** *problems are given in the back of the book.)*

We saw in Section 9–1 that the change in momentum is different for an object that hits something and sticks compared with an object that hits and bounces off. This means that the impulse, and hence the force, is different in the two cases. We explore this in the following Conceptual Checkpoint.

CONCEPTUAL CHECKPOINT 9–1 RAIN VERSUS HAIL

A person stands under an umbrella during a rain shower. A few minutes later the raindrops turn to hail—though the number of "drops" hitting the umbrella per time and their speed remain the same. Is the force required to hold the umbrella in the hail **(a)** the same as, **(b)** more than, or **(c)** less than the force required in the rain?



▲ Most bats can take off simply by dropping from their perch on a branch or the ceiling of a cave, but vampire bats like this one must leap from the ground to become airborne. They do so by rocking forward onto their front limbs and then pushing off, using the extremely strong pectoral muscles that are also their main source of power in flight. Pushing downward on the ground, a bat experiences an upward reaction force exerted on it by the ground, with a corresponding impulse sufficient to propel it upward a considerable distance. In fact, a vampire bat can launch itself 1 m or more into the air in a mere 30 ms.

REASONING AND DISCUSSION

When raindrops strike the umbrella, they tend to splatter and run off; when hailstones hit the umbrella, they bounce back upward. As a result, the change in momentum is greater for the hail—just as the change in momentum is greater for a rubber ball bouncing off the floor than it is for a beanbag landing on the floor. Hence, the impulse and the force are greater with hail.

ANSWER

(b) The force is greater in the hail.

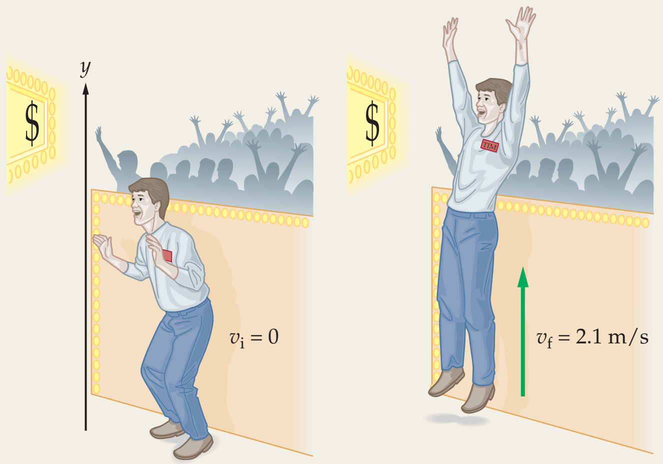
We conclude this section with an additional calculation involving impulse.

EXAMPLE 9–2 JUMPING FOR JOY

After winning a prize on a game show, a 72-kg contestant jumps for joy. **(a)** If the jump results in an upward speed of 2.1 m/s, what is the impulse experienced by the contestant? **(b)** Before the jump, the floor exerts an upward force of mg on the contestant. What additional average upward force does the floor exert if the contestant pushes down on it for 0.36 s during the jump?

PICTURE THE PROBLEM

Our sketch shows that the contestant's motion is purely one-dimensional, with a final speed of 2.1 m/s in the positive vertical direction. Note that we have chosen the positive y direction to be upward, therefore $\vec{\mathbf{v}}_i = 0$ and $\vec{\mathbf{v}}_f = (2.1\ \text{m/s})\hat{\mathbf{y}}$.



STRATEGY

a. From the momentum–impulse theorem, we know that impulse is equal to the change in momentum. We are given the initial and final velocities of the contestant, and his mass as well; hence the change in momentum, $\Delta\vec{\mathbf{p}}$, can be calculated using the definition of momentum, $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$.

b. The average value of the additional force exerted on the contestant by the floor is $\Delta\vec{\mathbf{p}}/\Delta t$, where Δt is given as 0.36 s and $\Delta\vec{\mathbf{p}}$ is calculated in part (a).

SOLUTION

Part (a)

1. Write an expression for the impulse, noting that $\vec{\mathbf{v}}_i = 0$: $\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = m\vec{\mathbf{v}}_f$

2. Substitute numerical values: $\vec{\mathbf{I}} = m\vec{\mathbf{v}}_f = (72\ \text{kg})(2.1\ \text{m/s})\hat{\mathbf{y}} = (150\ \text{kg}\cdot\text{m/s})\hat{\mathbf{y}}$

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Part (b)

3. Express the average force in terms of the impulse $\vec{\mathbf{I}}$ and the time interval Δt:

$$\vec{\mathbf{F}}_{\text{av}} = \frac{\vec{\mathbf{I}}}{\Delta t} = \frac{(150\ \text{kg}\cdot\text{m/s})\hat{\mathbf{y}}}{0.36\ \text{s}} = (420\ \text{kg}\cdot\text{m/s}^2)\hat{\mathbf{y}} = (420\ \text{N})\hat{\mathbf{y}}$$

INSIGHT

The magnitude of the additional average force exerted by the floor is rather large; in fact, 420 N is approximately 95 lb, or about 60% of the contestant's weight of 160 lb. Thus, the total upward force exerted by the floor is $mg + 420\ \text{N} = 710\ \text{N} + 420\ \text{N}$, which is about 250 lb. The contestant, of course, exerts the same force downward. Fortunately, the contestant only needs to exert that force for a third of a second.

When the contestant lands, an impulse is required to bring him to rest. If he lands with stiff legs, the impulse occurs in a short time, resulting in a large force delivered to the knees—with possible harmful effects. If he bends his legs on landing, on the other hand, the time duration is significantly increased, and the force applied to the contestant is correspondingly reduced.

PRACTICE PROBLEM

Suppose the contestant lands with a speed of 2.1 m/s and comes to rest in 0.25 s. What is the magnitude of the average force exerted by the floor during landing? [**Answer:** $mg + 600\ \text{N} \sim 290\ \text{lb}$]

Some related homework problems: Problem 13, Problem 14

9–4 Conservation of Linear Momentum

In this section we turn to perhaps the most significant aspect of linear momentum—the fact that it is a conserved quantity. In this respect, it plays a fundamental role in physics similar to that of energy. We shall also see that momentum conservation leads to calculational simplifications, making it of great practical significance.

First, recall that the net force acting on an object is equal to the rate of change of its momentum

$$\sum\vec{\mathbf{F}} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t}$$

Rearranging this expression, we find that the change in momentum during a time interval Δt is

$$\Delta\vec{\mathbf{p}} = \left(\sum\vec{\mathbf{F}}\right)\Delta t \qquad \text{9–7}$$

Clearly, then, if the net force acting on an object is zero,

$$\sum\vec{\mathbf{F}} = 0$$

its change in momentum is also zero:

$$\Delta\vec{\mathbf{p}} = \left(\sum\vec{\mathbf{F}}\right)\Delta t = 0$$

Writing the change of momentum in terms of its initial and final values, we have

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_{\text{f}} - \vec{\mathbf{p}}_{\text{i}} = 0$$

or

$$\vec{\mathbf{p}}_{\text{f}} = \vec{\mathbf{p}}_{\text{i}} \qquad \text{9–8}$$

Since the momentum does not change in this case, we say that it is **conserved.** To summarize:

Conservation of Momentum

If the net force acting on an object is zero, its momentum is conserved; that is,

$$\vec{\mathbf{p}}_{\text{f}} = \vec{\mathbf{p}}_{\text{i}}$$

Note that in some cases the force may be zero in one direction and nonzero in another. For example, an object in free fall has a nonzero y component of force, $F_y \neq 0$, but no force in the x direction, $F_x = 0$. As a result, the object's y component of momentum changes with time while its x component of momentum remains constant. Therefore, in applying momentum conservation, we must remember that both the force and the momentum are vector quantities and that the momentum conservation principle applies separately to each coordinate direction.

Thus far, our discussion has referred to the forces acting on a single object. Next, we consider a system composed of more than one object.

Internal Versus External Forces

The net force acting on a system of objects is the sum of forces applied from outside the system (external forces, $\vec{\mathbf{F}}_{\text{ext}}$) and forces acting between objects within the system (internal forces, $\vec{\mathbf{F}}_{\text{int}}$). Thus, we can write

$$\vec{\mathbf{F}}_{\text{net}} = \sum\vec{\mathbf{F}} = \sum\vec{\mathbf{F}}_{\text{ext}} + \sum\vec{\mathbf{F}}_{\text{int}}$$

As we shall see, internal and external forces play very *different* roles in terms of how they affect the momentum of a system.

To illustrate the distinction, consider the case of two canoes floating at rest next to one another on a lake, as described in Example 5–3 and shown in **Figure 9–4**. In this case, let's consider the "system" to be the two canoes and the people inside them. When a person in canoe 1 pushes on canoe 2, a force $\vec{\mathbf{F}}_2$ is exerted on canoe 2. By Newton's third law, an equal and opposite force, $\vec{\mathbf{F}}_1 = -\vec{\mathbf{F}}_2$, is exerted on the person in canoe 1. Note that $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are internal forces, since they act between objects in the system. In addition, note that they sum to zero:

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = (-\vec{\mathbf{F}}_2) + \vec{\mathbf{F}}_2 = 0$$

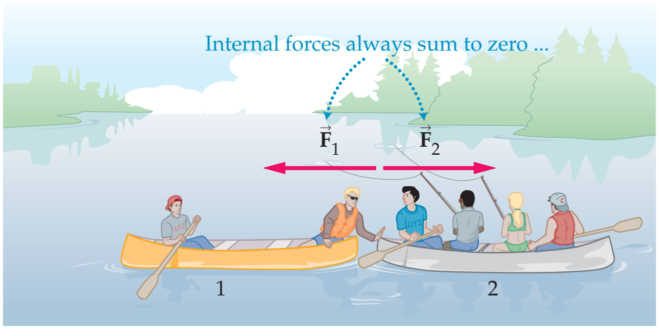


FIGURE 9–4 Separating two canoes
A system comprised of two canoes and their occupants. The forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are internal to the system. They sum to zero.

This is a special case, of course, but it demonstrates the following general principles:

- Internal forces, like all forces, always occur in action-reaction pairs.
- Because the forces in action-reaction pairs are equal and opposite—due to Newton's third law—internal forces must *always* sum to zero:

$$\sum\vec{\mathbf{F}}_{\text{int}} = 0$$

The fact that internal forces always cancel means that the net force acting on a system of objects is simply the sum of the *external* forces acting on it:

$$\vec{\mathbf{F}}_{\text{net}} = \sum\vec{\mathbf{F}}_{\text{ext}} + \sum\vec{\mathbf{F}}_{\text{int}} = \sum\vec{\mathbf{F}}_{\text{ext}}$$

The external forces, on the other hand, may or may not sum to zero—it all depends on the particular situation. For example, if the system consists of the two canoes in Figure 9–4, the external forces are the weights of the people and the canoes acting downward, and the upward, normal force exerted by the water to keep the canoes afloat. These forces sum to zero, and there is no acceleration in the vertical direction. In the next few sections we consider a variety of systems in which the external forces either sum to zero, or are so small that they can be ignored. Later, in Section 9–7, we consider situations where the external forces do not sum to zero and hence must be taken into account.

▲ If the astronaut in this photo pushes on the satellite, the satellite exerts an equal but opposite force on him, in accordance with Newton's third law. If we are calculating the change in the astronaut's momentum, we must take this force into account. However, if we define the system to be the astronaut *and* the satellite, the forces between them are internal to the system. Whatever effect they may have on the astronaut or the satellite individually, they do not affect the momentum of the system as a whole. Therefore, whether a particular force counts as internal or external depends entirely on where we draw the boundaries of the system.

Finally, how do external and internal forces affect the momentum of a system? To see the connection, first note that Newton's second law gives the change in the net momentum for a given time interval Δt:

$$\Delta\vec{\mathbf{p}}_{\text{net}} = \vec{\mathbf{F}}_{\text{net}}\,\Delta t$$

Because the internal forces cancel, however, the change in the net momentum is directly related to the net *external* force:

$$\Delta\vec{\mathbf{p}}_{\text{net}} = \left(\sum\vec{\mathbf{F}}_{\text{ext}}\right)\Delta t \qquad 9\text{–}9$$

Therefore, the key distinction between internal and external forces is the following:

Conservation of Momentum for a System of Objects

- *Internal* forces have absolutely no effect on the net momentum of a system.
- If the *net external* force acting on a system is zero, its net momentum is conserved. That is,

$$\vec{\mathbf{p}}_{1,\text{f}} + \vec{\mathbf{p}}_{2,\text{f}} + \vec{\mathbf{p}}_{3,\text{f}} + \cdots = \vec{\mathbf{p}}_{1,\text{i}} + \vec{\mathbf{p}}_{2,\text{i}} + \vec{\mathbf{p}}_{3,\text{i}} + \cdots$$

It is important to note that these statements apply only to the *net* momentum of a system, not to the momentum of each individual object. For example, suppose a system consists of two objects, 1 and 2, and that the net external force acting on the system is zero. As a result, the net momentum must remain constant:

$$\vec{\mathbf{p}}_{\text{net}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = \text{constant}$$

This does not mean, however, that $\vec{\mathbf{p}}_1$ is constant or that $\vec{\mathbf{p}}_2$ is constant. All we can say is that the *sum* of $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ does not change.

As a specific example, consider the case of the two canoes floating on a lake, as described previously. Initially the momentum of the system is zero, because the canoes are at rest. After a person pushes the canoes apart, they are both moving, and hence both have nonzero momentum. Thus, the momentum of each canoe has changed. On the other hand, because the net external force acting on the system is zero, the sum of the canoes' momenta must still vanish. We show this in the next Example.

PROBLEM-SOLVING NOTE

Internal Versus External Forces

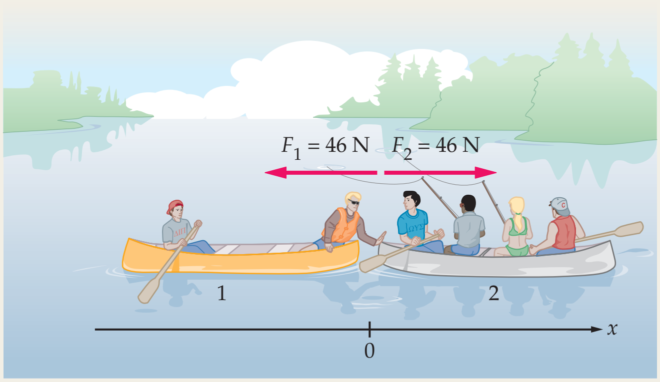
It is important to keep in mind that internal forces cannot change the momentum of a system—only a net external force can do that.

EXAMPLE 9–3 TIPPY CANOE: COMPARING VELOCITY AND MOMENTUM

Two groups of canoeists meet in the middle of a lake. After a brief visit, a person in canoe 1 pushes on canoe 2 with a force of 46 N to separate the canoes. If the mass of canoe 1 and its occupants is 130 kg, and the mass of canoe 2 and its occupants is 250 kg, find the momentum of each canoe after 1.20 s of pushing.

PICTURE THE PROBLEM

We choose the positive x direction to point from canoe 1 to canoe 2. With this choice, the force exerted on canoe 2 is $\vec{\mathbf{F}}_2 = (46\text{ N})\hat{\mathbf{x}}$ and the force exerted on canoe 1 is $\vec{\mathbf{F}}_1 = (-46\text{ N})\hat{\mathbf{x}}$.



STRATEGY

First, we find the acceleration of each canoe using $a_x = F_x/m$. Next, we use $v_x = v_{0x} + a_x t$ to find the velocity at time t. Note that the canoes start at rest, hence $v_{0x} = 0$. Finally, the momentum can be calculated using $p_x = mv_x$.

SOLUTION

1. Use Newton's second law to find the acceleration of canoe 2: $$a_{2,x} = \frac{\sum F_{2,x}}{m_2} = \frac{46\ \text{N}}{250\ \text{kg}} = 0.18\ \text{m/s}^2$$
2. Do the same calculation for canoe 1. Note that the acceleration of canoe 1 is in the negative direction: $$a_{1,x} = \frac{\sum F_{1,x}}{m_1} = \frac{-46\ \text{N}}{130\ \text{kg}} = -0.35\ \text{m/s}^2$$
3. Calculate the velocity of each canoe at $t = 1.20$ s: $$v_{1,x} = a_{1,x}t = (-0.35\ \text{m/s}^2)(1.20\ \text{s}) = -0.42\ \text{m/s}$$ $$v_{2,x} = a_{2,x}t = (0.18\ \text{m/s}^2)(1.20\ \text{s}) = 0.22\ \text{m/s}$$
4. Calculate the momentum of each canoe at $t = 1.20$ s: $$p_{1,x} = m_1 v_{1,x} = (130\ \text{kg})(-0.42\ \text{m/s}) = -55\ \text{kg}\cdot\text{m/s}$$ $$p_{2,x} = m_2 v_{2,x} = (250\ \text{kg})(0.22\ \text{m/s}) = 55\ \text{kg}\cdot\text{m/s}$$

INSIGHT

Note that the sum of the momenta of the two canoes is zero. This is just what one would expect: The canoes start at rest with zero momentum, there is zero net external force acting on the system, hence the final momentum must also be zero. The final velocities *do not* add to zero; it is momentum ($m\vec{\mathbf{v}}$) that is conserved, not velocity ($\vec{\mathbf{v}}$).

Finally, we solved this problem using one-dimensional kinematics so that we could clearly see the distinction between velocity and momentum. An alternative way to calculate the final momentum of each canoe is to use $\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_\text{f} - \vec{\mathbf{p}}_\text{i} = \vec{\mathbf{F}}\Delta t$. For canoe 1 we have $\vec{\mathbf{p}}_{1,\text{f}} = \vec{\mathbf{F}}_1\,\Delta t + \vec{\mathbf{p}}_{1,\text{i}} = (-46\ \text{N})\hat{\mathbf{x}}(1.20\ \text{s}) + 0 = (-55\ \text{kg}\cdot\text{m/s})\hat{\mathbf{x}}$, in agreement with our results above. A similar calculation yields $\vec{\mathbf{p}}_{2,\text{f}} = (55\ \text{kg}\cdot\text{m/s})\hat{\mathbf{x}}$ for canoe 2.

PRACTICE PROBLEM

What are the final momenta if the canoes are pushed apart with a force of 56 N? [**Answer:** $p_{1,x} = -67\ \text{kg}\cdot\text{m/s}$, $p_{2,x} = 67\ \text{kg}\cdot\text{m/s}$]

Some related homework problems: Problem 21, Problem 22

In a situation like that described in Example 9–3, the person in canoe 1 pushes canoe 2 away. At the same time, canoe 1 begins to move in the opposite direction. This is referred to as **recoil.** It is essentially the same as the recoil one experiences when firing a gun or when turning on a strong stream of water.

A particularly interesting example of recoil involves the human body. Perhaps you have noticed, when resting quietly in a rocking or reclining chair, that the chair wobbles back and forth slightly about once a second. The reason for this movement is that each time your heart pumps blood in one direction (from the atria to the ventricles, then to the aorta and pulmonary arteries, and so on) your body recoils in the opposite direction. Because the recoil depends on the force exerted by your heart on the blood and the volume of blood expelled from the heart with each beat, it is possible to gain valuable medical information regarding the health of your heart by analyzing the recoil it produces.

REAL-WORLD PHYSICS: BIO

The ballistocardiograph

The medical instrument that employs the physical principle of recoil is called the *ballistocardiograph*. It is a completely noninvasive technology that simply requires the patient to sit comfortably in a chair fitted with sensitive force sensors under the seat and behind the back. Sophisticated bathroom scales also utilize this technology. A ballistocardiographic (BCG) scale detects the recoil vibrations of the body as a person stands on the scale. This allows the BCG scale to display not only the person's body weight but his or her heart rate as well.

A more dramatic application of heartbeat recoil is currently being used at the Riverbend Maximum Security Institution in Tennessee. The only successful breakout from this prison occurred when four inmates hid in a secret compartment in a delivery truck that was leaving the facility. The institution now uses a heartbeat recoil detector that would have foiled this escape. Vehicles leaving the prison must stop at a checkpoint where a small motion detector is attached to it with a suction cup. Any persons hidden in the vehicle will reveal their presence by the very beating of their hearts. These heartbeat detectors have proved to be 100 percent effective, even though the recoil of the heart may displace a large truck by only a few millionths of an inch. Similar systems are being used at other high-security installations and border crossings.

REAL-WORLD PHYSICS: BIO

Heartbeat detectors

CONCEPTUAL CHECKPOINT 9–2 MOMENTUM VERSUS KINETIC ENERGY

In Example 9–3, the final momentum of the system (consisting of the two canoes and their occupants) is equal to the initial momentum of the system. Is the final kinetic energy **(a)** equal to, **(b)** less than, or **(c)** greater than the initial kinetic energy?

REASONING AND DISCUSSION

The final momentum of the two canoes is zero because one canoe has a positive momentum and the other has a negative momentum of the same magnitude. The two momenta, then, sum to zero. Kinetic energy, which is $\frac{1}{2}mv^2$, cannot be negative; hence no such cancellation is possible. Both canoes have positive kinetic energies, and therefore, the final kinetic energy is greater than the initial kinetic energy, which is zero.

Where does the increase in kinetic energy come from? It comes from the muscular work done by the person who pushes the canoes apart.

ANSWER

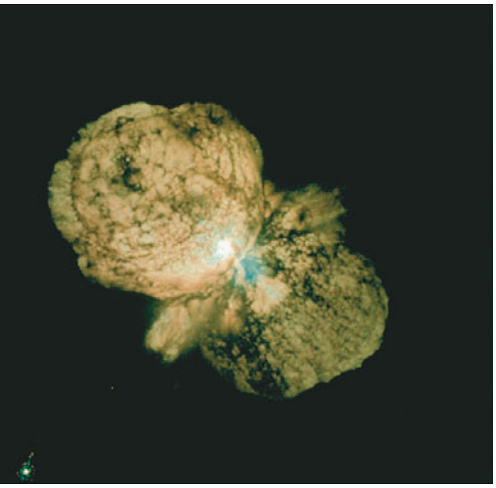
(c) K_f is greater than K_i.

REAL-WORLD PHYSICS

Stellar explosions

A special case of some interest is the universe. Since there is nothing external to the universe—by definition—it follows that the net external force acting on it is zero. Therefore, its net momentum is conserved. No matter what happens—a comet collides with the Earth, a star explodes and becomes a supernova, a black hole swallows part of a galaxy—the total momentum of the universe simply cannot change. A particularly vivid illustration of momentum conservation in our own galaxy is provided by the exploding star Eta Carinae. As can be seen in the Hubble Space Telescope photograph, jets of material are moving away from the star in opposite directions, just like the canoes moving apart from one another in Example 9–3.

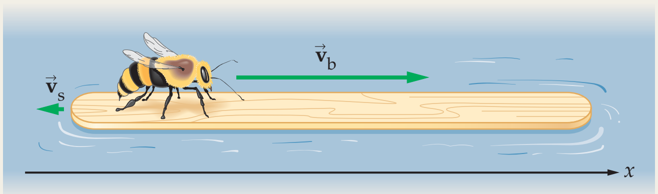
Conservation of momentum also applies to the more everyday situation described in the next Active Example.



▲ This Hubble Space Telescope photograph shows the aftermath of a violent explosion of the star Eta Carinae. The explosion, which was observed on Earth in 1841 and briefly made Eta Carinae the second brightest star in the sky, produced two bright lobes of matter spewing outward in opposite directions. In this photograph, these lobes have expanded to about the size of our solar system. The momentum of the star before the explosion must be the same as the total momentum of the star and the bright lobes after the explosion. Since the lobes are roughly symmetric and move in opposite directions, their net momentum is essentially zero. Thus, we conclude that the momentum of the star itself was virtually unchanged by the explosion.

ACTIVE EXAMPLE 9–2 FIND THE VELOCITY OF THE BEE

A honeybee with a mass of 0.150 g lands on one end of a floating 4.75-g popsicle stick. After sitting at rest for a moment, it runs toward the other end with a velocity $\vec{v}_b$ relative to the still water. The stick moves in the opposite direction with a speed of 0.120 cm/s. What is the velocity of the bee? (Let the direction of the bee's motion be the positive x direction.)



SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Set the total momentum of the system equal to zero: $\vec{p}_b + \vec{p}_s = 0$
2. Solve for the momentum of the bee: $\vec{p}_b = -\vec{p}_s = m_b v_b \hat{x}$
3. Calculate the momentum of the stick: $\vec{p}_s = -m_s v_s \hat{x} = (-0.570 \text{ g}\cdot\text{cm/s})\hat{x}$
4. Calculate the momentum of the bee: $\vec{p}_b = m_b v_b \hat{x} = -\vec{p}_s = (0.570 \text{ g}\cdot\text{cm/s})\hat{x}$
5. Divide by the bee's mass to find its velocity: $\vec{v}_b = \vec{p}_b/m_b = (3.80 \text{ cm/s})\hat{x}$

INSIGHT

Because only internal forces are at work while the bee walks on the stick, the system's total momentum must remain zero.

YOUR TURN

Suppose the mass of the popsicle stick is 9.50 g rather than 4.75 g. What is the bee's velocity in this case?

(Answers to **Your Turn** *problems are given in the back of the book.)*

9–5 Inelastic Collisions

We now turn our attention to **collisions.** By a collision we mean a situation in which two objects strike one another, and in which the net external force is either zero or negligibly small. For example, if two train cars roll along on a level track and hit one another, this is a collision. In this case, the net external force—the weight downward and the normal force exerted by the tracks upward—is zero. As a result, the momentum of the two-car system is conserved.

Another example of a collision is a baseball being struck by a bat. In this case, the external forces are not zero because the weight of the ball is not balanced by any other force. However, as we have seen in Section 9–3, the forces exerted during the hit are much larger than the weight of the ball or the bat. Hence, to a good approximation, we may neglect the external forces (the weight of the ball and bat) in this case, and say that the momentum of the ball–bat system is conserved.

Now it may seem surprising at first, but the fact that the momentum of a system is conserved during a collision does not necessarily mean that the system's kinetic energy is conserved. In fact most, or even all, of a system's kinetic energy may be converted to other forms during a collision while, at the same time, not one bit of momentum is lost. This shall be explored in detail in this section.

In general, collisions are categorized according to what happens to the kinetic energy of the system. There are two possibilities. After a collision, the final kinetic energy, K_f, is either equal to the initial kinetic energy, K_i, or it is not. If $K_f = K_i$, the collision is said to be **elastic.** We shall consider elastic collisions in the next section.

On the other hand, the kinetic energy may change during a collision. Usually it decreases due to losses associated with sound, heat, and deformation. Sometimes it increases, if the collision sets off an explosion, for instance. In any event, collisions in which the kinetic energy is not conserved are referred to as **inelastic:**

Inelastic Collisions

In an inelastic collision, the momentum of a system is conserved,

$$\vec{\mathbf{p}}_f = \vec{\mathbf{p}}_i$$

but its kinetic energy is not,

$$K_f \neq K_i$$

In both elastic and inelastic collisions, momentum is conserved. The same is not true of kinetic energy, however. In the largely inelastic collision at left, much of the hockey players' initial kinetic energy is transformed into work: rearranging the players' anatomies and shattering the glass of the rink. In the highly elastic collision at right, the ball rebounds with very little diminution of its kinetic energy (though a little energy is lost as sound and heat).

Finally, in the special case where objects stick together after the collision, we say that the collision is **completely inelastic.**

Completely Inelastic Collisions
When objects stick together after colliding, the collision is completely inelastic.

In a completely inelastic collision, the maximum amount of kinetic energy is lost. If the total momentum of the system is zero, this means that all of the kinetic energy is lost. For systems with nonzero total momentum, however, some kinetic energy will remain after the collision—still, the amount lost is the maximum permitted by momentum conservation.

Inelastic Collisions in One Dimension

Consider a system of two identical train cars of mass m on a smooth, level track. One car is at rest initially while the other moves toward it with a speed v_0, as shown in **Figure 9–5**. When the cars collide, the coupling mechanism latches, causing the cars to stick together and move as a unit. What is the speed of the cars after the collision?

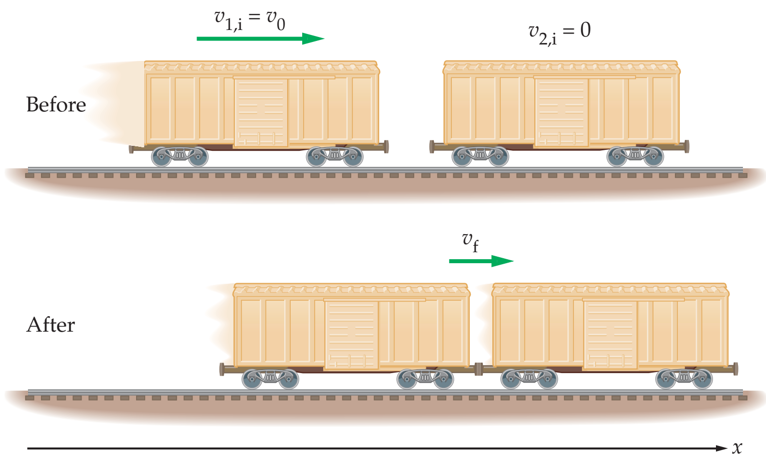


FIGURE 9–5 Railroad cars collide and stick together
A moving train car collides with an identical car that is stationary. After the collision, the cars stick together and move with the same speed.

To answer this question, we begin by considering the general case that applies to any completely inelastic collision, and then we look at the specific case of the two train cars. In general, suppose that two masses, m_1 and m_2, have initial velocities $v_{1,i}$ and $v_{2,i}$ respectively. The initial momentum of the system is

$$p_i = m_1 v_{1,i} + m_2 v_{2,i}$$

After the collision, the objects move together with a common velocity v_f. Therefore, the final momentum is

$$p_f = (m_1 + m_2)v_f$$

Equating the initial and final momenta yields $m_1 v_{1,i} + m_2 v_{2,i} = (m_1 + m_2)v_f$, or

$$v_f = \frac{m_1 v_{1,i} + m_2 v_{2,i}}{m_1 + m_2} \qquad 9\text{–}10$$

We can apply this general result to the case of the two railroad cars by noting that $m_1 = m_2 = m$, $v_{1,i} = v_0$, and $v_{2,i} = 0$. Thus, the final velocity is

$$v_f = \frac{mv_0 + m \cdot 0}{m + m} = \frac{m}{2m}v_0 = \frac{1}{2}v_0 \qquad 9\text{–}11$$

As you might have guessed, the final speed is one-half the initial speed.

EXERCISE 9–2

A 1200-kg car moving at 2.5 m/s is struck in the rear by a 2600-kg truck moving at 6.2 m/s. If the vehicles stick together after the collision, what is their speed immediately after colliding? (Assume that external forces may be ignored.)

SOLUTION

Applying Equation 9–10 with $m_1 = 1200$ kg, $v_{1,\mathrm{i}} = 2.5$ m/s, $m_2 = 2600$ kg, and $v_{2,\mathrm{i}} = 6.2$ m/s yields $v_\mathrm{f} = 5.0$ m/s.

During the collision of the railroad cars, some of the initial kinetic energy is converted to other forms. Some propagates away as sound, some is converted to heat, some creates permanent deformations in the metal of the latching mechanism. The precise amount of kinetic energy that is lost is addressed in the following Conceptual Checkpoint.

PROBLEM-SOLVING NOTE

Momentum Versus Energy Conservation

Be sure to distinguish between momentum conservation and energy conservation. A common error is to assume that kinetic energy is conserved just because the momentum is conserved.

CONCEPTUAL CHECKPOINT 9–3 HOW MUCH KINETIC ENERGY IS LOST?

A railroad car of mass m and speed v collides and sticks to an identical railroad car that is initially at rest. After the collision, is the kinetic energy of the system **(a)** 1/2, **(b)** 1/3, or **(c)** 1/4 of its initial kinetic energy?

REASONING AND DISCUSSION

Before the collision, the kinetic energy of the system is

$$K_\mathrm{i} = \frac{1}{2}mv^2$$

After the collision, the mass doubles and the speed is halved. Hence, the final kinetic energy is

$$K_\mathrm{f} = \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{2}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}K_\mathrm{i}$$

Therefore, one-half of the initial kinetic energy is converted to other forms of energy.

An equivalent way to arrive at this conclusion is to express the kinetic energy in terms of the momentum, $p = mv$:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{m^2v^2}{m}\right) = \frac{p^2}{2m}$$

Since the momentum is the same before and after the collision, the fact that the mass doubles means the kinetic energy is halved.

ANSWER

(a) The final kinetic energy is one-half the initial kinetic energy.

Note that we know the precise amount of kinetic energy that was lost, even though we don't know just how much went into sound, how much went into heat, and so on. It is not necessary to know all of those details to determine how much kinetic energy was lost.

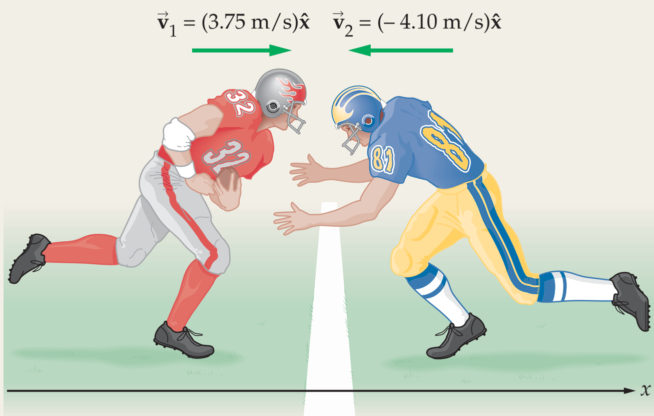
We also know how much momentum was lost—none.

EXAMPLE 9–4 GOAL-LINE STAND

On a touchdown attempt, a 95.0-kg running back runs toward the end zone at 3.75 m/s. A 111-kg linebacker moving at 4.10 m/s meets the runner in a head-on collision. If the two players stick together, **(a)** what is their velocity immediately after the collision? **(b)** What are the initial and final kinetic energies of the system?

PICTURE THE PROBLEM

In our sketch, we let subscript 1 refer to the red-and-gray running back, who carries the ball, and subscript 2 refer to the blue-and-gold linebacker, who will make the tackle. The direction of the running back's initial motion is taken to be in the positive x direction. Therefore, the initial velocities of the players are $\vec{\mathbf{v}}_1 = (3.75\ \mathrm{m/s})\hat{\mathbf{x}}$ and $\vec{\mathbf{v}}_2 = (-4.10\ \mathrm{m/s})\hat{\mathbf{x}}$.



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STRATEGY

a. The final velocity can be found by applying momentum conservation to the system consisting of the two players. Initially, the players have momenta in opposite directions. After the collision, the players move together with a combined mass $m_1 + m_2$ and a velocity $\vec{\mathbf{v}}_f$.

b. The kinetic energies can be found by applying $\frac{1}{2}mv^2$ to the players individually to obtain the initial kinetic energy, and then to their combined motion for the final kinetic energy.

SOLUTION

Part (a)

1. Set the initial momentum equal to the final momentum:

$$m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = (m_1 + m_2)\vec{\mathbf{v}}_f$$

2. Solve for the final velocity and substitute numerical values, being careful to use the appropriate signs:

$$\vec{\mathbf{v}}_f = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2}$$
$$= \frac{(95.0\ \text{kg})(3.75\ \text{m/s})\hat{\mathbf{x}} + (111\ \text{kg})(-4.10\ \text{m/s})\hat{\mathbf{x}}}{95.0\ \text{kg} + 111\ \text{kg}}$$
$$= (-0.480\ \text{m/s})\hat{\mathbf{x}}$$

Part (b)

3. Calculate the initial kinetic energy of the two players:

$$K_i = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$
$$= \tfrac{1}{2}(95.0\ \text{kg})(3.75\ \text{m/s})^2 + \tfrac{1}{2}(111\ \text{kg})(-4.10\ \text{m/s})^2$$
$$= 1600\ \text{J}$$

4. Calculate the final kinetic energy of the players, noting that they both move with the same velocity after the collision:

$$K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2$$
$$= \tfrac{1}{2}(95.0\ \text{kg} + 111\ \text{kg})(-0.480\ \text{m/s})^2 = 23.7\ \text{J}$$

INSIGHT

After the collision, the two players are moving in the negative direction; that is, away from the end zone. This is because the linebacker had more negative momentum than the running back had positive momentum. As for the kinetic energy, of the original 1600 J, only 23.7 J is left after the collision. This means that over 98% of the original kinetic energy is converted to other forms. Even so, *none* of the momentum is lost.

PRACTICE PROBLEM

If the final speed of the two players is to be zero, should the speed of the running back be increased or decreased? Check your answer by calculating the required speed for the running back. [**Answer:** The running back's speed should be increased to 4.79 m/s.]

Some related homework problems: Problem 28, Problem 35

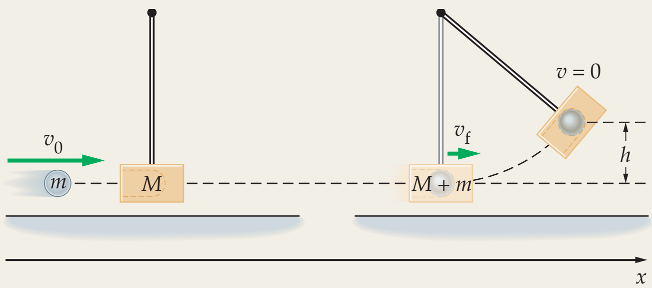
EXAMPLE 9–5 BALLISTIC PENDULUM

REAL-WORLD PHYSICS

In a ballistic pendulum, an object of mass m is fired with an initial speed v_0 at the bob of a pendulum. The bob has a mass M, and is suspended by a rod of negligible mass. After the collision, the object and the bob stick together and swing through an arc, eventually gaining a height h. Find the height h in terms of m, M, v_0, and g.

PICTURE THE PROBLEM

Our sketch shows the physical setup of a ballistic pendulum. Initially, only the object of mass m is moving, and it moves in the positive x direction with a speed v_0. Immediately after the collision, the bob and object move together with a new speed, v_f, which is determined by momentum conservation. Finally, the pendulum continues to swing to the right until its speed decreases to zero and it comes to rest at the height h.



INTERACTIVE FIGURE

STRATEGY

There are two distinct physical processes at work in the ballistic pendulum. The first is a completely inelastic collision between the bob and the object. Momentum is conserved during this collision, but kinetic energy is not. After the collision, the remaining kinetic energy is converted into gravitational potential energy, which determines how high the bob and object will rise.

SOLUTION

1. Set the momentum just before the bob–object collision equal to the momentum just after the collision. Let v_f be the speed just after the collision:

$$mv_0 = (M + m)v_f$$

2. Solve for the speed just after the collision, v_f:

$$v_f = \left(\frac{m}{M + m}\right)v_0$$

3. Calculate the kinetic energy just after the collision:

$$K_f = \frac{1}{2}(M + m)v_f^2 = \frac{1}{2}(M + m)\left(\frac{m}{M + m}\right)^2 v_0^2$$
$$= \frac{1}{2}mv_0^2\left(\frac{m}{M + m}\right)$$

4. Set the kinetic energy after the collision equal to the gravitational potential energy at the height h:

$$\frac{1}{2}mv_0^2\left(\frac{m}{M + m}\right) = (M + m)gh$$

5. Solve for the height, h:

$$h = \left(\frac{m}{M + m}\right)^2\left(\frac{v_0^2}{2g}\right)$$

INSIGHT

A ballistic pendulum is often used to measure the speed of a rapidly moving object, such as a bullet. If a bullet were shot straight up, it would rise to the height $v_0^2/2g$, which can be thousands of feet. On the other hand, if a bullet of mass m is fired into a ballistic pendulum, in which M is much greater than m, the bullet reaches only a small fraction of this height. Thus, the ballistic pendulum makes for a more convenient and practical measurement.

PRACTICE PROBLEM

A 7.00-g bullet is fired into a ballistic pendulum whose bob has a mass of 0.950 kg. If the bob rises to a height of 0.220 m, what was the initial speed of the bullet? [**Answer:** $v_0 = 284$ m/s. If this bullet were fired straight up, it would rise 4.11 km $\approx$ 13,000 ft in the absence of air resistance.]

Some related homework problems: Problem 32, Problem 33

Inelastic Collisions in Two Dimensions

Next we consider collisions in two dimensions, where we must conserve the momentum component by component. To do this, we set up a coordinate system and resolve the initial momentum into x and y components. Next, we demand that the final momentum have precisely the same x and y components as the initial momentum. That is,

$$p_{x,i} = p_{x,f}$$

and

$$p_{y,i} = p_{y,f}$$

The following Example shows how to carry out such a calculation in a practical situation.

PROBLEM-SOLVING NOTE

Sketch the System Before and After the Collision

In problems involving collisions, it is useful to draw the system before and after the collision. Be sure to label the relevant masses, velocities, and angles.

EXAMPLE 9–6 BAD INTERSECTION: ANALYZING A TRAFFIC ACCIDENT

REAL-WORLD PHYSICS

A car with a mass of 950 kg and a speed of 16 m/s approaches an intersection, as shown on the next page. A 1300-kg minivan traveling at 21 m/s is heading for the same intersection. The car and minivan collide and stick together. Find the speed and direction of the wrecked vehicles just after the collision, assuming external forces can be ignored.

PICTURE THE PROBLEM

In our sketch, we align the x and y axes with the crossing streets. With this choice, $\vec{\mathbf{v}}_1$ (the car's velocity) is in the positive x direction, and $\vec{\mathbf{v}}_2$ (the minivan's velocity) is in the positive y direction. In addition, the problem statement indicates that

CONTINUED ON NEXT PAGE

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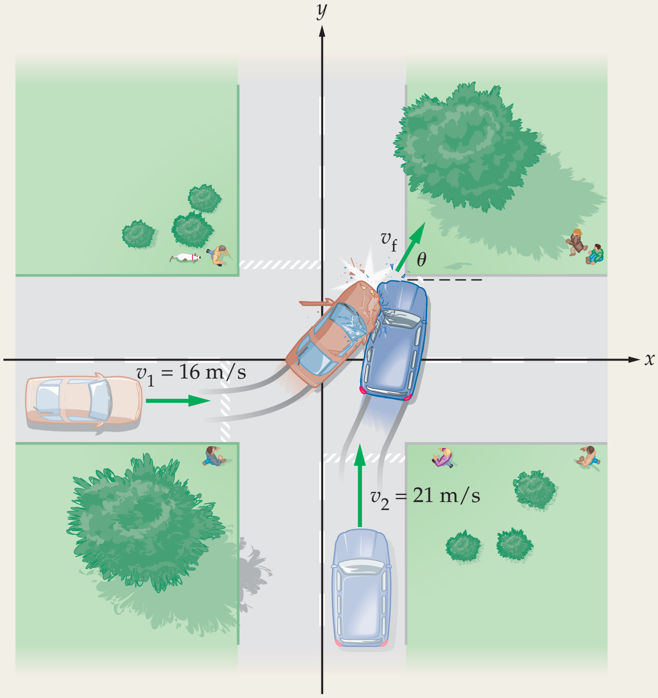
$m_1 = 950$ kg and $m_2 = 1300$ kg. After the collision, the two vehicles move together (as a unit) with a speed v_f in a direction θ with respect to the positive x axis.



STRATEGY

Because external forces can be ignored, the total momentum of the system must be conserved during the collision. This is really two conditions: (i) the x component of momentum is conserved, and (ii) the y component of momentum is conserved. These two conditions determine the two unknowns: the final speed, v_f, and the final direction, θ.

SOLUTION

1. Set the initial x component of momentum equal to the final x component of momentum:

$$m_1 v_1 = (m_1 + m_2) v_f \cos\theta$$

2. Do the same for the y component of momentum:

$$m_2 v_2 = (m_1 + m_2) v_f \sin\theta$$

3. Divide the y momentum equation by the x momentum equation. This eliminates v_f, giving an equation involving θ alone:

$$\frac{m_2 v_2}{m_1 v_1} = \frac{(m_1 + m_2) v_f \sin\theta}{(m_1 + m_2) v_f \cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

4. Solve for θ:

$$\theta = \tan^{-1}\left(\frac{m_2 v_2}{m_1 v_1}\right) = \tan^{-1}\left[\frac{(1300\text{ kg})(21\text{ m/s})}{(950\text{ kg})(16\text{ m/s})}\right]$$

$$= \tan^{-1}(1.8) = 61^\circ$$

5. The final speed can be found using either the x or the y momentum equation. Here we use the x equation:

$$v_f = \frac{m_1 v_1}{(m_1 + m_2)\cos\theta}$$

$$= \frac{(950\text{ kg})(16\text{ m/s})}{(950\text{ kg} + 1300\text{ kg})\cos 61^\circ} = 14\text{ m/s}$$

INSIGHT

As a check, you should verify that the y momentum equation gives the same value for v_f.

When a collision occurs in the real world, a traffic-accident investigation team will measure skid marks at the scene of the crash and use this information—along with some basic physics—to determine the initial speeds and directions of the vehicles. This information is often presented in court, where it can lead to a clear identification of the driver at fault.

PRACTICE PROBLEM

Suppose the speed and direction immediately after the collision are known to be $v_f = 12.5$ m/s and $\theta = 42^\circ$, respectively. Find the initial speed of each car. [**Answer:** $v_1 = 22$ m/s, $v_2 = 14$ m/s]

Some related homework problems: Problem 29, Problem 30

9–6 Elastic Collisions

In this section we consider collisions in which both momentum and kinetic energy are conserved. As mentioned in the previous section, such collisions are referred to as elastic:

Elastic Collisions

In an elastic collision, momentum and kinetic energy are conserved. That is,

$$\vec{\mathbf{p}}_f = \vec{\mathbf{p}}_i$$

and

$$K_f = K_i$$

Most collisions in everyday life are rather poor approximations to being elastic—usually there is a significant amount of energy converted to other forms. However, the collision of objects that bounce off one another with little deformation—like billiard balls, for example—provides a reasonably good approximation to an elastic collision. In the subatomic world, on the other hand, elastic collisions are common. Elastic collisions, then, are not merely an ideal that is approached but never attained—they are constantly taking place in nature.

Elastic Collisions in One Dimension

Consider a head-on collision of two carts on an air track, as pictured in **Figure 9–6**. The carts are provided with bumpers that give an elastic bounce when the carts collide. Let's suppose that initially cart 1 is moving to the right with a speed v_0 toward cart 2, which is at rest. If the masses of the carts are m_1 and m_2, respectively, then momentum conservation can be written as follows:

$$m_1 v_0 = m_1 v_{1,f} + m_2 v_{2,f}$$

In this expression, $v_{1,f}$ and $v_{2,f}$ are the final velocities of the two carts. Note that we say velocities, not speeds, since it is possible for cart 1 to reverse direction, in which case $v_{1,f}$ would be negative.

Next, the fact that this is an elastic collision means the final velocities must also satisfy energy conservation:

$$\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

Thus, we now have two equations for the two unknowns, $v_{1,f}$ and $v_{2,f}$. Straightforward—though messy—algebra yields the following results:

$$v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_0$$
$$v_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_0 \qquad \text{9–12}$$

Note that the final velocity of cart 1 can be positive, negative, or zero, depending on whether m_1 is greater than, less than, or equal to m_2, respectively. The final velocity of cart 2, however, is always positive.

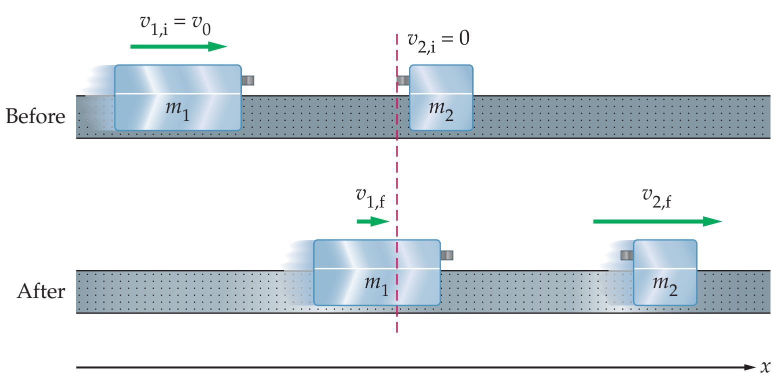


FIGURE 9–6 An elastic collision between two air carts

In the case pictured, $v_{1,f}$ is to the right (positive), which means that m_1 is greater than m_2. In fact, we have chosen $m_1 = 2m_2$ for this plot; therefore, $v_{1,f} = v_0/3$ and $v_{2,f} = 4v_0/3$ as given by Equations 9–12. If m_1 were less than m_2, cart 1 would bounce back toward the left, meaning that $v_{1,f}$ would be negative.

EXERCISE 9–3

At an amusement park, a 96.0-kg bumper car moving with a speed of 1.24 m/s bounces elastically off a 135-kg bumper car at rest. Find the final velocities of the cars.

SOLUTION

Using Equations 9–12, we find the final velocities to be $v_{1,\mathrm{f}} = -0.209\ \mathrm{m/s}$ and $v_{2,\mathrm{f}} = 1.03\ \mathrm{m/s}$. Note that the direction of travel of car 1 has been reversed.

Let's check a few special cases of our results. First, consider the case where the two carts have equal masses, $m_1 = m_2 = m$. Substituting into Equations 9–12, we find

$$v_{1,\mathrm{f}} = \left(\frac{m - m}{m + m}\right)v_0 = 0$$

and

$$v_{2,\mathrm{f}} = \left(\frac{2m}{m + m}\right)v_0 = v_0$$

Thus, after the collision, the cart that was moving with velocity v_0 is now at rest, and the cart that was at rest is now moving with velocity v_0. In effect, the carts have "exchanged" velocities. This case is illustrated in **Figure 9–7 (a)**.

Next, suppose that m_2 is much greater than m_1, or, equivalently, that m_1 approaches zero. Returning to Equations 9–12, and setting $m_1 = 0$, we find

$$v_{1,\mathrm{f}} = \left(\frac{0 - m_2}{0 + m_2}\right)v_0 = \left(\frac{-m_2}{m_2}\right)v_0 = -v_0$$

and

$$v_{2,\mathrm{f}} = \frac{2 \cdot 0}{0 + m_2}v_0 = 0$$

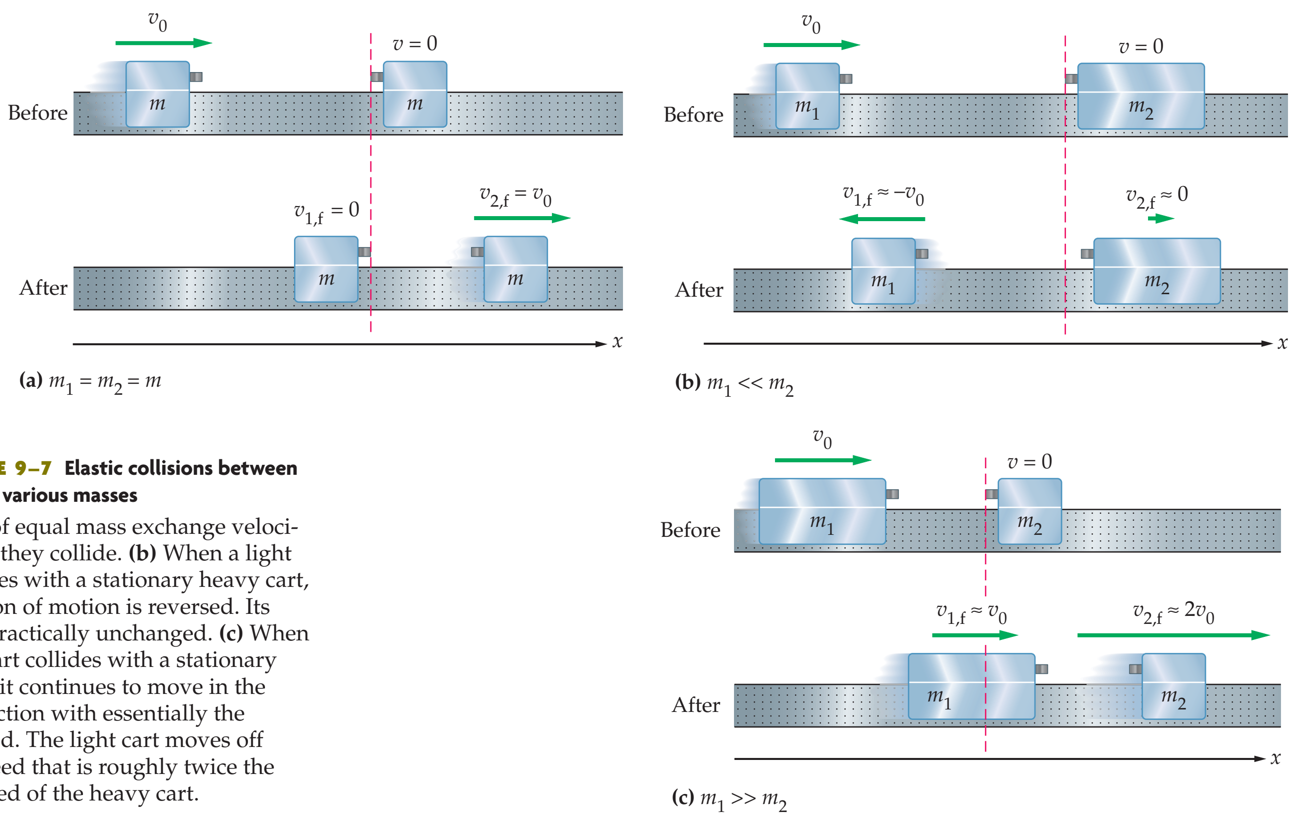
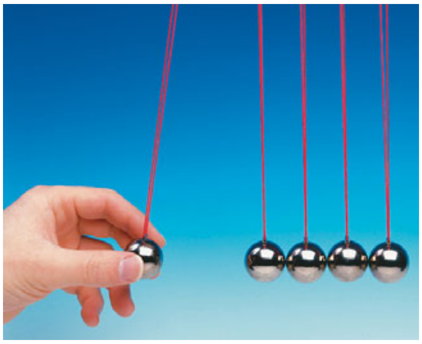


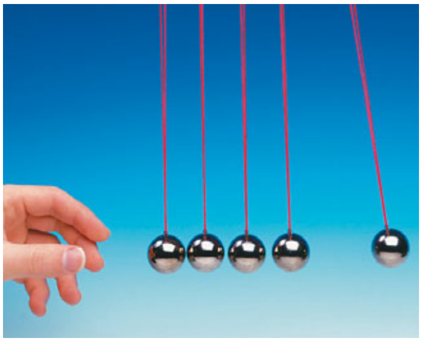
FIGURE 9–7 Elastic collisions between air carts of various masses

(a) Carts of equal mass exchange velocities when they collide. **(b)** When a light cart collides with a stationary heavy cart, its direction of motion is reversed. Its speed is practically unchanged. **(c)** When a heavy cart collides with a stationary light cart, it continues to move in the same direction with essentially the same speed. The light cart moves off with a speed that is roughly twice the initial speed of the heavy cart.

Physically, we interpret these results as follows: A very light cart collides with a heavy cart that is at rest. The heavy cart hardly budges, but the light cart is reflected, heading *backward* (remember the minus sign in $-v_0$) with the same speed it had initially. For example, if you throw a ball against a wall, the wall is the very heavy object and the ball is the light object. The ball bounces back with the same speed it had initially (assuming an ideal elastic collision). We show a case in which m_1 is much less than m_2 in **Figure 9–7 (b)**.

Finally, what happens when m_1 is much greater than m_2? To check this limit we can set m_2 equal to zero. We consider the results in the following Conceptual Checkpoint.





The apparatus shown here illustrates some of the basic features of elastic collisions between objects of equal mass. The device consists of five identical metal balls suspended by strings. When the end ball is pulled out to the side and then released so as to fall back and strike the second ball, it creates a rapid succession of elastic collisions among the balls. In each collision, one ball comes to rest while the next one begins to move with the original speed, just as with the air carts in Figure 9–7 (a). When the collisions reach the other end of the apparatus, the last ball swings out to the same height from which the first ball was released.

If two balls are pulled out and released, two balls swing out at the other side, and so on. To see why this must be so, imagine that the two balls swing in with a speed v and a single ball swings out at the other side with a speed v'. What value must v' have (a) to conserve momentum, and (b) to conserve kinetic energy? Since the required speed is $v' = 2v$ for (a) and $v' = \sqrt{2}v$ for (b), it follows that it is not possible to conserve both momentum and kinetic energy with two balls swinging in and one ball swinging out.

CONCEPTUAL CHECKPOINT 9–4 SPEED AFTER A COLLISION

A hoverfly is happily maintaining a fixed position about 10 ft above the ground when an elephant charges out of the bush and collides with it. The fly bounces elastically off the forehead of the elephant. If the initial speed of the elephant is v_0, is the speed of the fly after the collision equal to **(a)** v_0, **(b)** $1.5v_0$, or **(c)** $2v_0$?

REASONING AND DISCUSSION

We can use Equations 9–12 to find the final speeds of the fly and the elephant. First, let m_1 be the mass of the elephant, and m_2 be the mass of the fly. Clearly, m_2 is vanishingly small compared with m_1, hence we can evaluate Equations 9–12 in the limit $m_2 \to 0$. This yields

$$v_{1,\mathrm{f}} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_0 \xrightarrow[m_2 \to 0]{} \left(\frac{m_1}{m_1}\right)v_0 = v_0$$

and

$$v_{2,\mathrm{f}} = \left(\frac{2m_1}{m_1 + m_2}\right)v_0 \xrightarrow[m_2 \to 0]{} \left(\frac{2m_1}{m_1}\right)v_0 = 2v_0$$

As expected, the speed of the elephant is unaffected. The fly, however, rebounds with twice the speed of the elephant. **Figure 9–7 (c)** illustrates this case with air carts.

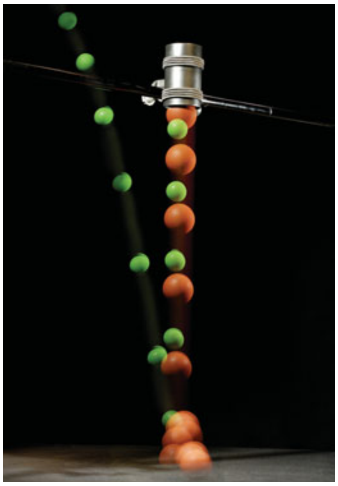
ANSWER

(c) The speed of the fly is $2v_0$.

Note that after the collision the fly is separating from the elephant with the speed $2v_0 - v_0 = v_0$. Before the collision the elephant was approaching the fly with the same speed, v_0. This is a special case of the following general result:

> The speed of separation after a head-on elastic collision is always equal to the speed of approach before the collision.

The proof of this statement is the subject of Problem 91.



Momentum Transfer and Height Amplification
In a collision between two objects of different mass, like the small and large balls in this photo, a significant amount of momentum can be transferred from the large object to the small object. Even though the total momentum is conserved, the small object can be given a speed that is significantly larger than any of the initial speeds. This is illustrated in the photo by the height to which the small ball bounces. A similar process occurs in the collapse of a star during a supernova explosion. The resulting collision can send jets of material racing away from the supernova at nearly the speed of light, just like the small ball that takes off with such a large speed in this collision.

Elastic Collisions in Two Dimensions

In a two-dimensional elastic collision, if we are given the final speed and direction of one of the objects, we can find the speed and direction of the other object using energy conservation and momentum conservation. For example, consider the collision of two 7.00-kg curling stones, as depicted in **Figure 9–8**. One stone is at rest initially, the other approaches with a speed $v_{1,i} = 1.50\ \mathrm{m/s}$. The collision is not head-on, and after the collision, stone 1 moves with a speed of $v_{1,f} = 0.610\ \mathrm{m/s}$ in a direction 66.0° away from the initial line of motion. What are the speed and direction of stone 2?

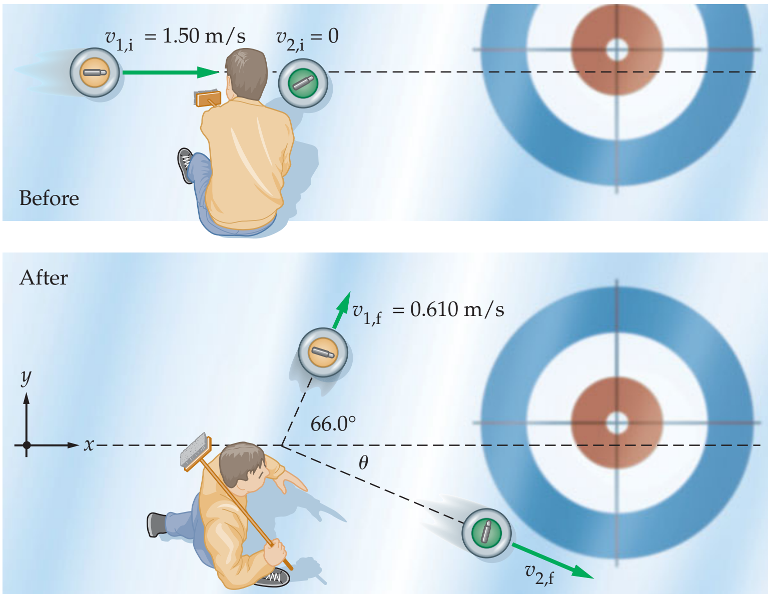


FIGURE 9–8 Two curling stones undergo an elastic collision

The speed of curling stone 2 after this collision can be determined using energy conservation; its direction of motion can be found using momentum conservation in either the x or the y direction.

PROBLEM-SOLVING NOTE

Kinetic Energy in Elastic Collisions

Remember that in elastic collisions, by definition, the kinetic energy is conserved.

First, let's find the speed of stone 2. The easiest way to do this is to simply require that the final kinetic energy be equal to the initial kinetic energy. Initially, the kinetic energy is

$$K_i = \tfrac{1}{2}m_1{v_{1,i}}^2 = \tfrac{1}{2}(7.00\ \mathrm{kg})(1.50\ \mathrm{m/s})^2 = 7.88\ \mathrm{J}$$

After the collision stone 1 has a speed of 0.610 m/s and stone 2 has the speed $v_{2,f}$. Hence, the final kinetic energy is

$$\begin{aligned} K_f &= \tfrac{1}{2}m_1{v_{1,f}}^2 + \tfrac{1}{2}m_2{v_{2,f}}^2 = \tfrac{1}{2}(7.00\ \mathrm{kg})(0.610\ \mathrm{m/s})^2 + \tfrac{1}{2}m_2{v_{2,f}}^2 \\ &= 1.30\ \mathrm{J} + \tfrac{1}{2}m_2{v_{2,f}}^2 = K_i \end{aligned}$$

Solving for the speed of stone 2, we find

$$v_{2,f} = 1.37\ \mathrm{m/s}$$

Next, we can find the direction of motion of stone 2 by requiring that the momentum be conserved. For example, initially there is no momentum in the y direction. This must be true after the collision as well. Hence, we have the following condition:

$$0 = m_1v_{1,f}\sin 66.0° - m_2v_{2,f}\sin\theta$$

Solving for the angle θ we find

$$\theta = 24.0°$$

As a final check, compare the initial and final x components of momentum. Initially, we have

$$p_{x,i} = m_1v_{1,i} = (7.00\ \mathrm{kg})(1.50\ \mathrm{m/s}) = 10.5\ \mathrm{kg\cdot m/s}$$

Following the collision, the x component of momentum is

$$p_{x,\mathrm{f}} = m_1 v_{1,\mathrm{f}} \cos 66.0° + m_2 v_{2,\mathrm{f}} \cos 24.0°$$
$$= (7.00\ \mathrm{kg})(0.610\ \mathrm{m/s}) \cos 66.0° + (7.00\ \mathrm{kg})(1.37\ \mathrm{m/s}) \cos 24.0°$$
$$= 10.5\ \mathrm{kg \cdot m/s}$$

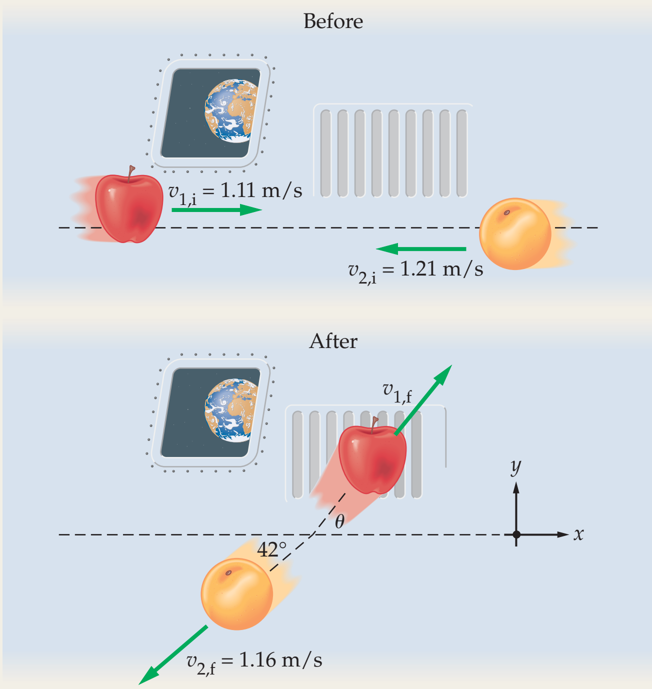
As expected, the momentum is unchanged.

EXAMPLE 9–7 TWO FRUITS IN TWO DIMENSIONS: ANALYZING AN ELASTIC COLLISION

Two astronauts on opposite ends of a spaceship are comparing lunches. One has an apple, the other has an orange. They decide to trade. Astronaut 1 tosses the 0.130-kg apple toward astronaut 2 with a speed of 1.11 m/s. The 0.160-kg orange is tossed from astronaut 2 to astronaut 1 with a speed of 1.21 m/s. Unfortunately, the fruits collide, sending the orange off with a speed of 1.16 m/s at an angle of 42.0° with respect to its original direction of motion. Find the final speed and direction of the apple, assuming an elastic collision. Give the apple's direction relative to its original direction of motion.

PICTURE THE PROBLEM

In our sketch we refer to the apple as object 1 and to the orange as object 2. We also choose the positive x direction to be in the initial direction of motion of the apple. We shall describe the "Before" and "After" sketches separately:



BEFORE

Initially, the apple moves in the positive x direction with a speed of 1.11 m/s, and the orange moves in the negative x direction with a speed of 1.21 m/s. There is no momentum in the y direction before the collision.

AFTER

After the collision, the orange moves with a speed of 1.16 m/s in a direction 42° below the negative x axis. As a result, the orange now has momentum in the negative y direction. To cancel this y momentum, the apple must move in a direction that is above the positive x axis, as indicated in the sketch.

STRATEGY

As described in the text, we first find the speed of the apple by demanding that the initial and final kinetic energies be the same. Next, we find the angle θ by conserving momentum in either the x or the y direction—the results are the same whichever direction is chosen.

SOLUTION

1. Calculate the initial kinetic energy of the system:

$$K_\mathrm{i} = \tfrac{1}{2} m_1 v_{1,\mathrm{i}}^2 + \tfrac{1}{2} m_2 v_{2,\mathrm{i}}^2$$
$$= \tfrac{1}{2}(0.130\ \mathrm{kg})(1.11\ \mathrm{m/s})^2 + \tfrac{1}{2}(0.160\ \mathrm{kg})(1.21\ \mathrm{m/s})^2$$
$$= 0.197\ \mathrm{J}$$

2. Calculate the final kinetic energy of the system in terms of $v_{1,\mathrm{f}}$:

$$K_\mathrm{f} = \tfrac{1}{2} m_1 v_{1,\mathrm{f}}^2 + \tfrac{1}{2} m_2 v_{2,\mathrm{f}}^2$$
$$= \tfrac{1}{2}(0.130\ \mathrm{kg}) v_{1,\mathrm{f}}^2 + \tfrac{1}{2}(0.160\ \mathrm{kg})(1.16\ \mathrm{m/s})^2$$
$$= \tfrac{1}{2}(0.130\ \mathrm{kg}) v_{1,\mathrm{f}}^2 + 0.108\ \mathrm{J}$$

3. Set $K_\mathrm{f} = K_\mathrm{i}$ to find $v_{1,\mathrm{f}}$:

$$v_{1,\mathrm{f}} = \sqrt{\frac{2(0.197\ \mathrm{J} - 0.108\ \mathrm{J})}{0.130\ \mathrm{kg}}} = 1.17\ \mathrm{m/s}$$

4. Set the final y component of momentum equal to zero to determine the angle, θ:

$$0 = m_1 v_{1,\mathrm{f}} \sin\theta - m_2 v_{2,\mathrm{f}} \sin 42.0°$$

Solve for $\sin\theta$:

$$\sin\theta = \frac{m_2 v_{2,\mathrm{f}} \sin 42.0°}{m_1 v_{1,\mathrm{f}}}$$

5. Substitute numerical values:

$$\sin\theta = \frac{(0.160\ \mathrm{kg})(1.16\ \mathrm{m/s}) \sin 42.0°}{(0.130\ \mathrm{kg})(1.17\ \mathrm{m/s})} = 0.817$$

$$\theta = \sin^{-1}(0.817) = 54.8°$$

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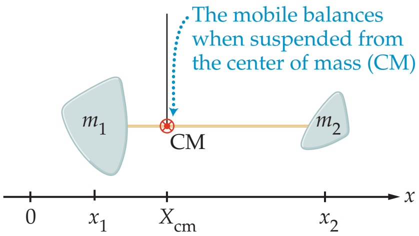
INSIGHT

The x momentum equation gives the same value for θ, as expected.

PRACTICE PROBLEM

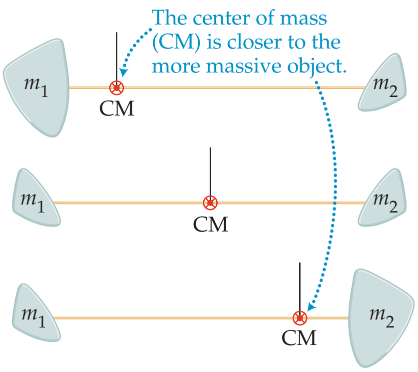
Suppose that after the collision the apple moves in the positive y direction with a speed of 1.27 m/s. What are the final speed and direction of the orange in this case? [**Answer:** The orange moves with a speed of 1.07 m/s in a direction of 74.7° below the negative x axis.]

Some related homework problems: Problem 41, Problem 94



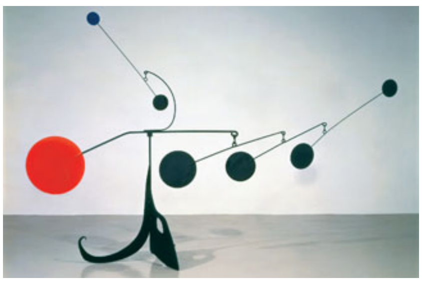
▲ **FIGURE 9–9 Balancing a mobile**

Consider a portion of a mobile with masses m_1 and m_2 at the locations x_1 and x_2, respectively. The object balances when a string is attached at the center of mass. Since the center of mass is closer to m_1 than to m_2, it follows that m_1 is greater than m_2.



▲ **FIGURE 9–10 The center of mass of two objects**

The center of mass is closest to the larger mass, or equidistant between the masses if they are equal.



▲ Mobiles like *Myxomatose* by Alexander Calder illustrate the concept of center of mass with artistic flair. Each arm of the mobile is in balance because it is suspended at its center of mass.

9–7 Center of Mass

In this section we introduce the concept of the center of mass. We begin by defining its location for a given system of masses. Next we consider the motion of the center of mass and show how it is related to the net external force acting on the system. As we shall see, the center of mass plays a key role in the analysis of collisions.

Location of the Center of Mass

There is one point in any system of objects that has special significance—the **center of mass (CM).** One of the reasons the center of mass is so special is the fact that, in many ways, a system behaves as if all of its mass were concentrated there. As a result, a system can be balanced at its center of mass:

> The center of mass of a system of masses is the point where the system can be balanced in a uniform gravitational field.

For example, suppose you are making a mobile. At one stage in its construction, you want to balance a light rod with objects of mass m_1 and m_2 connected to either end, as indicated in **Figure 9–9**. To make the rod balance, you should attach a string to the center of mass of the system, just as if all its mass were concentrated at that point. In a sense, you can think of the center of mass as the "average" location of the system's mass.

To be more specific, suppose the two objects connected to the rod have the same mass. In this case the center of mass is at the midpoint of the rod, since this is where it balances. On the other hand, if one object has more mass than the other, the center of mass is closer to the heavier object, as indicated in **Figure 9–10**. In general, if a mass m_1 is on the x axis at the position x_1, and a mass m_2 is at the position x_2, as in Figure 9–9, the location of the center of mass, X_{cm}, is defined as the "weighted" average of the two positions:

Center of Mass for Two Objects

$$X_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1x_1 + m_2x_2}{M} \qquad 9\text{–}13$$

Note that we have used $M = m_1 + m_2$ for the total mass of the two objects, and that the two positions, x_1 and x_2, are multiplied—or weighted—by their respective masses.

To see that this definition of X_{cm} agrees with our expectations, consider first the case where the masses are equal: $m_1 = m_2 = m$. In this case, $M = m_1 + m_2 = 2m$, and $X_{cm} = (mx_1 + mx_2)/2m = \frac{1}{2}(x_1 + x_2)$. Thus, as expected, if two masses are equal, their center of mass is halfway between them. On the other hand, if m_1 is significantly greater than m_2, it follows that $M = m_1 + m_2 \sim m_1$ and $m_1x_1 + m_2x_2 \sim m_1x_1$, since m_2 can be ignored in comparison to m_1. As a result, we find that $X_{cm} \sim m_1x_1/m_1 = x_1$; that is, the center of mass is essentially at the location of the extremely heavy mass, m_1. In general, as one mass becomes larger than the other, the center of mass moves closer to the larger mass.

EXERCISE 9–4

Suppose the masses in Figure 9–9 are separated by 0.500 m, and that $m_1 = 0.260$ kg and $m_2 = 0.170$ kg. What is the distance from m_1 to the center of mass of the system?

SOLUTION

Letting $x_1 = 0$ and $x_2 = 0.500$ m in Figure 9–9, we have

$$X_{\text{cm}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{(0.260\text{ kg})\cdot 0 + (0.170\text{ kg})(0.500\text{ m})}{0.260\text{ kg} + 0.170\text{ kg}} = 0.198\text{ m}$$

Thus, the center of mass is closer to m_1 (the larger mass) than to m_2.

To extend the definition of X_{cm} to more general situations, first consider a system that contains many objects, not just two. In that case, X_{cm} is the sum of m times x for each object, divided by the total mass of the system, M. If, in addition, the objects in the system are not in a line, but are distributed in two dimensions, the center of mass will have both an x coordinate, X_{cm}, and a y coordinate, Y_{cm}. As one would expect, Y_{cm} is simply the sum of m times y for each object, divided by M. Thus, the x coordinate of the center of mass is

X Coordinate of the Center of Mass

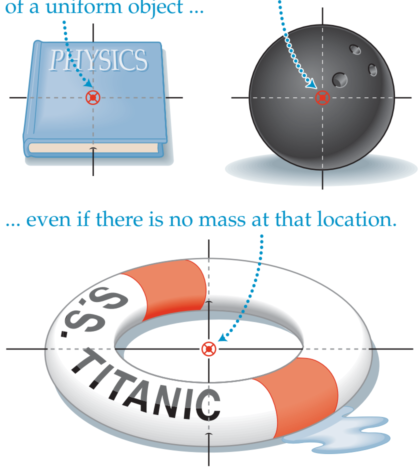
$$X_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum mx}{M} \qquad \text{9–14}$$

Similarly, the y coordinate of the center of mass is

Y Coordinate of the Center of Mass

$$Y_{\text{cm}} = \frac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum my}{M} \qquad \text{9–15}$$

In systems with a continuous, uniform distribution of mass, the center of mass is at the geometric center of the object, as illustrated in **Figure 9–11**. Note that it is common for the center of mass to be located in a position where no mass exists, as in a life preserver, where the center of mass is precisely in the center of the hole.



▲ **FIGURE 9–11 Locating the center of mass**

In an object of continuous, uniform mass distribution, the center of mass is located at the geometric center of the object. In some cases, this means that the center of mass is not located within the object.

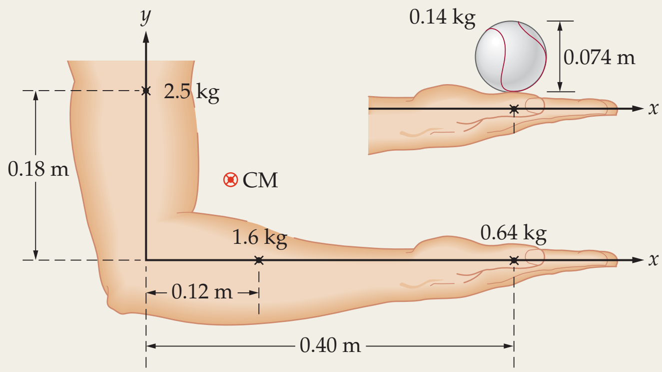
EXAMPLE 9–8 CENTER OF MASS OF THE ARM

REAL-WORLD PHYSICS: BIO

A person's arm is held with the upper arm vertical, the lower arm and hand horizontal. **(a)** Find the center of mass of the arm in this configuration, given the following data: The upper arm has a mass of 2.5 kg and a center of mass 0.18 m above the elbow; the lower arm has a mass of 1.6 kg and a center of mass 0.12 m to the right of the elbow; the hand has a mass of 0.64 kg and a center of mass 0.40 m to the right of the elbow. **(b)** A 0.14-kg baseball is placed on the palm of the hand. If the diameter of the ball is 7.4 cm, find the center of mass of the arm–ball system.

PICTURE THE PROBLEM

We place the origin at the elbow, with the x and y axes pointing along the lower and upper arms, respectively. The center of mass of each of the three parts of the arm is indicated by an x; the center of mass of the entire arm is at the point labeled CM. The inset shows the baseball on the palm of the hand.



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STRATEGY

a. Using the information given in the problem statement, we can treat the arm as a system of three point masses placed as follows: 2.5 kg at (0, 0.18 m); 1.6 kg at (0.12 m, 0); 0.64 kg at (0.40 m, 0). We substitute these masses and locations into Equations 9–14 and 9–15 to find the x and y coordinates of the center of mass, respectively.

b. Treat the center of mass found in part (a) as a point particle with a mass 2.5 kg + 1.6 kg + 0.64 kg = 4.7 kg at the location (X_{cm}, Y_{cm}). The baseball can be treated as a point particle of mass 0.14 kg at the location (0.40 m, (0.074)/2 m).

SOLUTION

Part (a)

1. Calculate the x coordinate of the center of mass:

$$X_{cm} = \frac{(2.5\text{ kg})(0) + (1.6\text{ kg})(0.12\text{ m}) + (0.64\text{ kg})(0.40\text{ m})}{2.5\text{ kg} + 1.6\text{ kg} + 0.64\text{ kg}} = 0.095\text{ m}$$

2. Do the same calculation for the y coordinate of the center of mass:

$$Y_{cm} = \frac{(2.5\text{ kg})(0.18\text{ m}) + (1.6\text{ kg})(0) + (0.64\text{ kg})(0)}{2.5\text{ kg} + 1.6\text{ kg} + 0.64\text{ kg}} = 0.095\text{ m}$$

Part (b)

3. Calculate the new x coordinate of the center of mass:

$$X_{cm} = \frac{(4.7\text{ kg})(0.095\text{ m}) + (0.14\text{ kg})(0.40\text{ m})}{4.7\text{ kg} + 0.14\text{ kg}} = 0.10\text{ m}$$

4. Calculate the new y coordinate of the center of mass:

$$Y_{cm} = \frac{(4.7\text{ kg})(0.095\text{ m}) + (0.14\text{ kg})(0.037\text{ m})}{4.7\text{ kg} + 0.14\text{ kg}} = 0.093\text{ m}$$

INSIGHT

As is often the case, the center of mass of an arm held in this position is in a location where no mass exists—you might say the center of mass is having an out-of-body experience. This effect can sometimes be put to good use, as when the center of mass of a high jumper passes beneath the horizontal bar while the body passes above it. See Conceptual Question 18 for a photo of this technique in action, in the famous "Fosbury flop."

PRACTICE PROBLEM

Suppose the mass of the baseball is increased to 0.25 kg. **(a)** Does X_{cm} increase, decrease, or stay the same? **(b)** Does Y_{cm} increase, decrease, or stay the same? **(c)** Check your answers to parts (a) and (b) by finding the center of mass of the arm–ball system in this case. [**Answer:** (a) increases; (b) decreases; (c) $X_{cm} = 0.11$ m, $Y_{cm} = 0.092$ m]

Some related homework problems: Problem 51, Problem 53

Motion of the Center of Mass

Another reason the center of mass is of such importance is that its motion often displays a remarkable simplicity when compared with the motion of other parts of a system. To analyze this motion, we consider both the velocity and the acceleration of the center of mass. Each of these quantities is defined in complete analogy with the definition of the center of mass itself.

For example, to find the velocity of the center of mass, we first multiply the mass of each object in a system, m, by its velocity, $\vec{\mathbf{v}}$, to give $m_1\vec{\mathbf{v}}_1$, $m_2\vec{\mathbf{v}}_2$, and so on. Next, we add all these products together, $m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots$, and divide by the total mass, $M = m_1 + m_2 + \cdots$. The result, by definition, is the velocity of the center of mass, $\vec{\mathbf{V}}_{cm}$:

Velocity of the Center of Mass

$$\vec{\mathbf{V}}_{cm} = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m\vec{\mathbf{v}}}{M} \qquad 9\text{–}16$$

Comparing with Equation 9–14, we see that $\vec{\mathbf{V}}_{cm}$ is the same as X_{cm} with each position x replaced with a velocity vector $\vec{\mathbf{v}}$. In addition, note that the total mass of the system, M, times the velocity of the center of mass, $\vec{\mathbf{V}}_{\text{cm}}$, is simply the total momentum of the system:

$$M\vec{\mathbf{V}}_{\text{cm}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots = \vec{\mathbf{p}}_{\text{total}}$$

To gain more information on how the center of mass moves, we next consider its acceleration, $\vec{\mathbf{A}}_{\text{cm}}$. As expected by analogy with $\vec{\mathbf{V}}_{\text{cm}}$, the acceleration of the center of mass is defined as follows:

Acceleration of the Center of Mass

$$\vec{\mathbf{A}}_{\text{cm}} = \frac{m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m\vec{\mathbf{a}}}{M} \qquad \text{9–17}$$

Note that the vector $\vec{\mathbf{A}}_{\text{cm}}$ contains terms like $m_1\vec{\mathbf{a}}_1$, $m_2\vec{\mathbf{a}}_2$, and so on, for each object in the system. From Newton's second law, however, we know that $m_1\vec{\mathbf{a}}_1$ is simply $\vec{\mathbf{F}}_1$, the net force acting on mass 1. The same conclusion applies to each of the masses. Therefore, we find that the total mass of the system, M, times the acceleration of the center of mass, $\vec{\mathbf{A}}_{\text{cm}}$, is simply the total force acting on the system:

$$M\vec{\mathbf{A}}_{\text{cm}} = m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{F}}_{\text{total}}$$

Recall, however, that the total force acting on a system is the same as the net external force, $\vec{\mathbf{F}}_{\text{net,ext}}$, since the internal forces cancel. Therefore, $M\vec{\mathbf{A}}_{\text{cm}}$ is the net external force acting on the system:

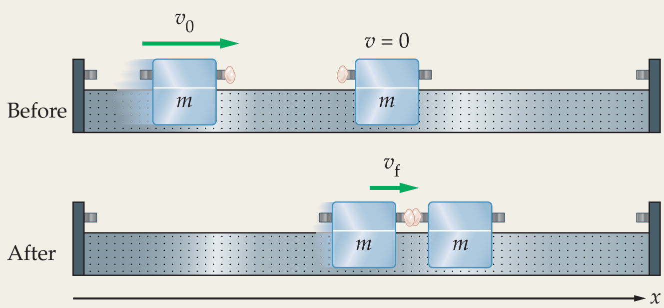
Newton's Second Law for a System of Particles

$$M\vec{\mathbf{A}}_{\text{cm}} = \vec{\mathbf{F}}_{\text{net,ext}} \qquad \text{9–18}$$

Zero Net External Force For systems in which $\vec{\mathbf{F}}_{\text{net,ext}}$ is zero, it follows that the acceleration of the center of mass is zero. Hence, if the center of mass is initially at rest, it remains at rest. Similarly, if the center of mass is moving initially, it continues to move with the same velocity. For example, in a collision between two air-track carts, the velocity of each cart changes as a result of the collision. The velocity of the center of mass of the two carts, however, is the same before and after the collision. We explore cases in which $\vec{\mathbf{F}}_{\text{net,ext}} = 0$ in the following Example and Active Example.

EXAMPLE 9–9 CRASH OF THE AIR CARTS

An air cart of mass m and speed v_0 moves toward a second, identical air cart that is at rest. When the carts collide they stick together and move as one. Find the velocity of the center of mass of this system **(a)** before and **(b)** after the carts collide.



INTERACTIVE FIGURE

PICTURE THE PROBLEM

We choose the positive x direction to be the direction of motion of the incoming cart, whose initial speed is v_0. Note that the carts have wads of putty on their bumpers; this ensures that they stick together when they collide and thereafter move as a unit. Their final speed is v_f.

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STRATEGY

a. We can find the velocity of the center of mass by applying Equation 9–16 to the case of just two masses; $\vec{\mathbf{V}}_{cm} = (m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2)/M$. In this case, $\vec{\mathbf{v}}_1 = v_0\hat{\mathbf{x}}$, $\vec{\mathbf{v}}_2 = 0$, and $m_1 = m_2 = m$.

b. After the collision the two masses have the same velocity, $\vec{\mathbf{v}}_f = v_f\hat{\mathbf{x}}$, which is given by momentum conservation (Equations 9–10 and 9–11). Hence, $\vec{\mathbf{V}}_{cm} = (m_1\vec{\mathbf{v}}_f + m_2\vec{\mathbf{v}}_f)/M$.

SOLUTION

Part (a)

1. Use $\vec{\mathbf{V}}_{cm} = (m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2)/M$ to find the velocity of the center of mass before the collision:

$$\vec{\mathbf{V}}_{cm} = \frac{(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2)}{m_1 + m_2} = \frac{(mv_0\hat{\mathbf{x}} + m \cdot 0)}{m + m} = \tfrac{1}{2}v_0\hat{\mathbf{x}}$$

Part (b)

2. Use momentum conservation in the x direction to find the speed of the carts after the collision:

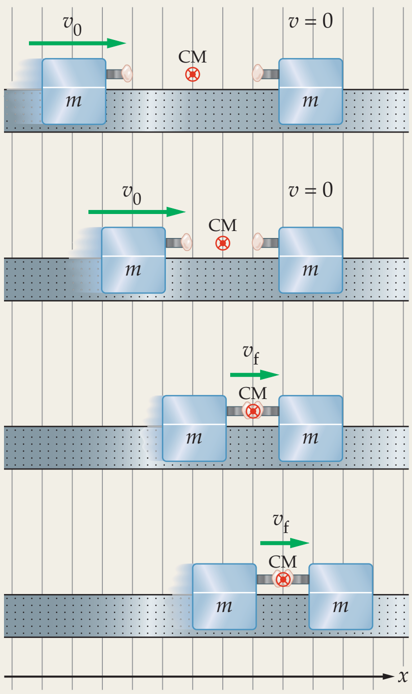
$$mv_0 = mv_f + mv_f$$
$$v_f = \tfrac{1}{2}v_0$$

3. Calculate the velocity of the center of mass of the two carts after the collision:

$$\vec{\mathbf{V}}_{cm} = \frac{(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2)}{m_1 + m_2} = \frac{(mv_f\hat{\mathbf{x}} + mv_f\hat{\mathbf{x}})}{m + m} = v_f\hat{\mathbf{x}} = \tfrac{1}{2}v_0\hat{\mathbf{x}}$$

INSIGHT

As expected, the velocity of each cart changes when they collide. On the other hand, the velocity of the center of mass is completely unaffected by the collision. This is illustrated to the right, where we show a sequence of equal-time snapshots of the system just before and just after the collision. First, we note that the incoming cart moves two distance units for every time interval until it collides with the second cart. From that point on, the two carts are locked together, and move one distance unit per time interval. In contrast, the center of mass (CM), which is centered between the two equal-mass carts, progresses uniformly throughout the sequence, advancing one unit of distance for each time interval.



PRACTICE PROBLEM

If the mass of the cart that is moving initially is doubled to $2m$, does the velocity of the center of mass increase, decrease, or stay the same? Verify your answer by calculating the velocity of the center of mass in this case. [**Answer:** The velocity of the center of mass increases. We find that $\vec{\mathbf{V}}_{cm} = (2v_0/3)\hat{\mathbf{x}}$, both before and after the collision.]

Some related homework problems: Problem 54, Problem 57, Problem 81

ACTIVE EXAMPLE 9–3 FIND THE VELOCITY OF THE CENTER OF MASS

In Active Example 9–2 we found that as a 0.150-g bee runs with a speed of 3.80 cm/s in one direction, the 4.75-g popsicle stick on which it floats moves with a speed of 0.120 cm/s in the opposite direction. Find the velocity of the center of mass of the bee and the stick.

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Write the velocity of the bee: $\vec{\mathbf{v}}_b = (3.80\text{ cm/s})\hat{\mathbf{x}}$

2. Write the velocity of the stick: $\vec{\mathbf{v}}_s = (-0.120\text{ cm/s})\hat{\mathbf{x}}$

3. Use these velocities to calculate $\vec{\mathbf{V}}_{cm}$: $\vec{\mathbf{V}}_{cm} = (m_b\vec{\mathbf{v}}_b + m_s\vec{\mathbf{v}}_s)/(m_b + m_s) = 0$

INSIGHT

$\vec{\mathbf{V}}_{cm}$ is zero, and hence the center of mass stays at rest as the bee and the stick move. This is as expected, since the net external force is zero for this system, and the bee and stick started at rest initially.

YOUR TURN

If the bee increases its speed, will the velocity of the center of mass be nonzero?

(Answers to **Your Turn** *problems are given in the back of the book.)*

Nonzero Net External Force Recall that Newton's second law, as expressed in Equation 9–18, states that the acceleration of the center of mass is related to the net external force as follows:

$$M\vec{\mathbf{A}}_{cm} = \vec{\mathbf{F}}_{net,ext}$$

This is completely analogous to the relationship between the acceleration of an object of mass m and the net force $\vec{\mathbf{F}}_{net}$ applied to it:

$$m\vec{\mathbf{a}} = \vec{\mathbf{F}}_{net}$$

Therefore, when $\vec{\mathbf{F}}_{net,ext}$ is nonzero, we can conclude the following:

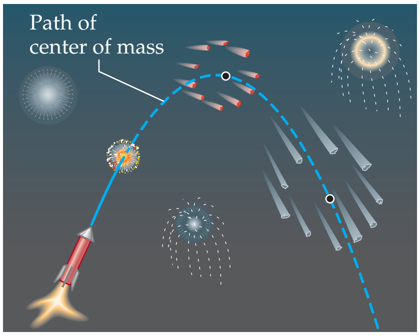
> The center of mass of a system accelerates precisely as if it were a point particle of mass M acted on by the force $\vec{\mathbf{F}}_{net,ext}$.

For this reason, the motion of the center of mass can be quite simple compared to the motion of its constituent parts. For example, a hammer tossed into the air with a rotation is shown in **Figure 9–12**. The motion of one part of the hammer, the tip of the handle, let's say, follows a complicated path in space. On the other hand, the path of the center of mass is a simple parabola, precisely the same path that a point mass would follow.

Similarly, consider a fireworks rocket launched into the sky, as illustrated in **Figure 9–13**. The center of mass of the rocket follows a parabolic path, ignoring air resistance. At some point in its path it explodes into numerous individual pieces. The explosion is due to internal forces, however, which must therefore sum to zero. Hence, the net external force acting on the pieces of the rocket is the same before, during, and after the explosion. As a result, the center of mass has a constant downward acceleration and continues to follow the original parabolic path. It is only when an additional external force acts on the system, as when one of the pieces of the rocket hits the ground, that the path of the center of mass changes.

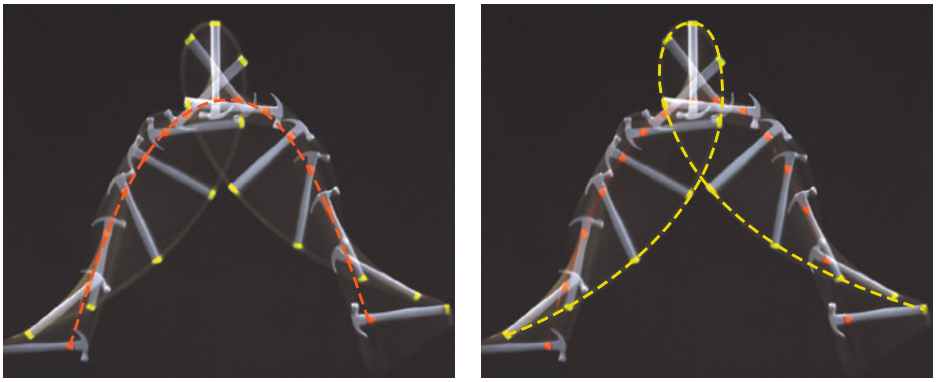
REAL-WORLD PHYSICS

An exploding rocket



▲ **FIGURE 9–13 Center of mass of an exploding rocket**

A fireworks rocket follows a parabolic path, ignoring air resistance, until it explodes. After exploding, its center of mass continues on the same parabolic path until some of the fragments start to land.



▲ **FIGURE 9–12 Simple Motion of the Center of Mass**

As this hammer flies through the air, its motion is quite complex. Some parts of the hammer follow wild trajectories with strange loops and turns. There is one point on the hammer, however, that travels on a smooth, simple parabolic path—the center of mass. The center of mass (red path on the left) travels as if all the mass of the hammer were concentrated there; other points (yellow path on the right) follow complex paths that depend on the detailed shape and rotation of the hammer.

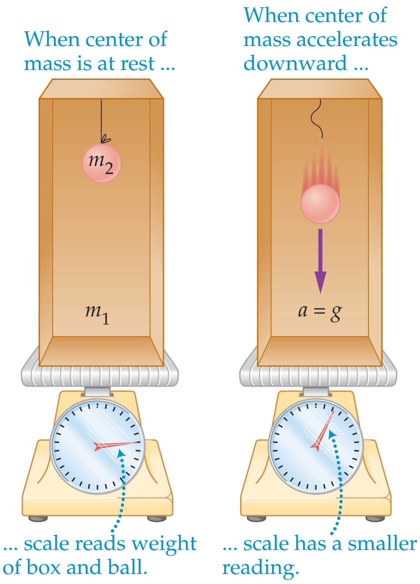


FIGURE 9–14 Weight and acceleration of the center of mass

A box with a ball suspended from a string is weighed on a scale. The scale reads the weight of the box and the ball. When the string breaks and the ball falls with the acceleration of gravity, the scale reads only the weight of the box.

To see how to apply $M\vec{\mathbf{A}}_{cm} = \vec{\mathbf{F}}_{net,ext}$, consider the system shown in **Figure 9–14**. Here we see a box of mass m_1, inside of which is a ball of mass m_2 suspended from a light string. The entire system rests on a scale reading its weight. The scale exerts an upward force on the box of magnitude F_s. Initially, of course, $F_s = (m_1 + m_2)g$.

Now, suppose the string breaks, allowing the ball to fall with constant acceleration g toward the bottom of the box. What is the reading on the scale while the ball falls? We can guess that the answer should be simply $m_1 g$, the weight of the box alone, but let's analyze the problem from the point of view of the center of mass.

Taking upward as the positive direction, the net external force acting on the box and the ball is

$$F_{net,ext} = F_s - m_1 g - m_2 g$$

The acceleration of the center of mass is

$$A_{cm} = \frac{m_1 \cdot 0 - m_2 g}{M} = -\frac{m_2}{M}g$$

Setting $MA_{cm} = F_{net,ext}$ yields

$$MA_{cm} = M\left(-\frac{m_2}{M}\right)g = -m_2 g = F_{net,ext} = F_s - m_1 g - m_2 g$$

Finally, canceling the term $-m_2 g$ and solving for the weight read by the scale, F_s, we find, as expected, that

$$F_s = m_1 g$$

*9–8 Systems with Changing Mass: Rocket Propulsion

We close this chapter by considering systems in which the mass can change. A rocket, for example, changes its mass as its engines operate because it ejects part of the fuel as it burns. The burning process is produced by internal forces, hence the total momentum of the rocket and its fuel remains constant.

Consider, then, a rocket in outer space, far from any large, massive objects. When the rocket's engine is fired, it expels a certain mass of fuel out the back with a speed v. If the mass of the ejected fuel is Δm, then the momentum of the ejected fuel has a magnitude equal to $(\Delta m)v$. Since the total momentum of the system must still be zero, the rocket acquires an equivalent amount of momentum in the forward direction. Hence, the momentum increase of the rocket is

$$\Delta p = (\Delta m)v$$

If the mass of fuel Δm is ejected in the time Δt, the force exerted on the rocket is the change in its momentum divided by the time interval (Equation 9–3); that is,

$$F = \frac{\Delta p}{\Delta t} = \left(\frac{\Delta m}{\Delta t}\right)v$$

The force exerted on the rocket by the ejected fuel is referred to as the **thrust.** Thus, the thrust of a rocket is

Thrust

$$\text{thrust} = \left(\frac{\Delta m}{\Delta t}\right)v \qquad 9\text{–}19$$

SI unit: newton, N

By $\Delta m/\Delta t$, we simply mean the amount of mass per time coming out of the rocket. For example, on the Saturn V rocket, the one used on the manned missions to the Moon, the main engines eject fuel at the rate of 13,800 kg/s with a speed of 2440 m/s. As a result, the thrust produced by these engines is

$$\text{thrust} = \left(\frac{\Delta m}{\Delta t}\right)v = (13{,}800\ \text{kg/s})(2440\ \text{m/s}) = 33.7 \times 10^6\ \text{N}$$

Since this is about 7.60 million pounds, and the weight of the rocket at liftoff is only 6.30 million pounds $= 28.0 \times 10^6$ N, the thrust is sufficient to launch the rocket and give it an upward acceleration. In fact, the initial net force acting on the rocket is

$$F_{\text{net}} = \text{thrust} - mg = 33.7 \times 10^6\ \text{N} - 28.0 \times 10^6\ \text{N} = 5.7 \times 10^6\ \text{N}$$

The rocket's initial weight is $W = mg = 28.0 \times 10^6$ N, and hence its initial mass is $m = W/g = 2.85 \times 10^6$ kg. Therefore, the rocket lifts off with an upward acceleration of

$$a = \frac{F_{\text{net}}}{m} = \frac{5.7 \times 10^6\ \text{N}}{2.85 \times 10^6\ \text{kg}} = 2.0\ \text{m/s}^2 \approx 0.20g$$

This is a rather gentle acceleration. The gentleness lasts only a matter of seconds, however, since the decreasing mass of the rocket results in an increasing acceleration.

REAL-WORLD PHYSICS
Saturn V rocket

EXERCISE 9–5

The ascent stage of the lunar lander was designed to produce 15,500 N of thrust at liftoff. If the speed of the ejected fuel is 2500 m/s, what is the rate at which the fuel must be burned?

SOLUTION

The rate of fuel consumption is

$$\frac{\Delta m}{\Delta t} = \frac{\text{thrust}}{v} = \frac{15{,}500\ \text{N}}{2500\ \text{m/s}} = 6.2\ \text{kg/s}$$

A common question regarding rockets is: "How can a rocket accelerate in outer space when it has nothing to push against?" The answer is that rockets, in effect, push against their own fuel. The situation is similar to firing a gun. When a bullet is ejected by the internal combustion of the gunpowder, the person firing the gun feels a recoil. If the person were in space, or standing on a frictionless surface, the recoil would give him or her a speed in the direction opposite to the bullet. The burning of a rocket engine provides a continuous recoil, almost as if the rocket were firing a steady stream of bullets out the back.

▲ A rocket (top) makes use of the principle of conservation of momentum: mass (the products of explosive burning of fuel) is ejected at high speed in one direction, causing the rocket to move in the opposite direction. The same method of propulsion has evolved in octopi (bottom) and some other animals. When danger threatens and a quick escape is needed, powerful muscles contract to create a jet of water that propels the animal to safety.

THE BIG PICTURE PUTTING PHYSICS IN CONTEXT

LOOKING BACK

We see in Section 9–1 that momentum is a vector quantity. Thus, the vector tools introduced in Chapter 3 again become important. In particular, we use vector components in our analysis of momentum conservation in Sections 9–5 and 9–6.

The connection between force (Chapter 5) and momentum is developed in Section 9–2. Newton's second law is central to impulse (Section 9–3), and Newton's third law is the key to conservation of momentum (Section 9–4).

Kinetic energy (Chapter 7) plays a key role in analyzing collisions, leading to the distinction between elastic and inelastic collisions. Potential energy (Chapter 8) enters into our analysis of the ballistic pendulum in Example 9–5.

In this chapter we see that force times the time over which it acts is related to a change in energy; in Chapters 7 and 8 we saw that force times the distance over which it acts leads to a change in energy.

LOOKING AHEAD

The concept of momentum is used again in Chapter 11, when we study the dynamics of rotational motion. In particular, we introduce angular momentum in Section 11–6 as an extension of the linear momentum introduced in this chapter.

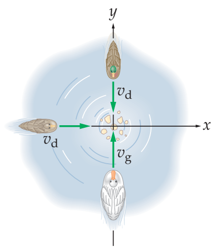
Angular momentum is used in our analysis of planetary orbits, and especially in the discussion of Kepler's second law in Section 12–3.

The idea of angular momentum having only certain allowed values is one of the key assumptions of the Bohr model of the hydrogen atom, as we show in Section 31–3.

Linear momentum plays an important role in quantum physics. For example, the momentum of a particle is related to its de Broglie wavelength (Section 30–5) and to the uncertainty principle (Section 30–6).

CHAPTER SUMMARY

9–1 LINEAR MOMENTUM



The linear momentum of an object of mass m moving with velocity $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \qquad \text{9–1}$$

Momentum Is a Vector
Linear momentum is a vector, pointing in the same direction as the velocity vector, $\vec{\mathbf{v}}$.

Momentum of a System of Objects
In a system of several objects, the total linear momentum is the vector sum of the individual momenta:

$$\vec{\mathbf{p}}_{\text{total}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3 + \cdots \qquad \text{9–2}$$

9–2 MOMENTUM AND NEWTON'S SECOND LAW

In terms of momentum, Newton's second law is

$$\sum \vec{\mathbf{F}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t} \qquad \text{9–3}$$

That is, the net force acting on an object is equal to the rate of change of its momentum.

Constant Mass
For cases in which the mass is constant, Newton's second law reduces to the familiar form

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}} \qquad \text{9–4}$$

9–3 IMPULSE

The impulse delivered to an object by an average force $\vec{\mathbf{F}}_{\text{av}}$ acting for a time Δt is

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\text{av}}\,\Delta t \qquad \text{9–5}$$

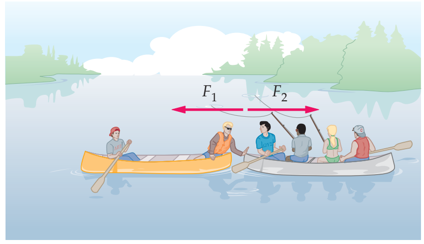
Impulse Is a Vector
Impulse is a vector, proportional to the force vector.

Impulse and Momentum
By Newton's second law, the impulse delivered to an object is equal to the change in its momentum:

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\text{av}}\,\Delta t = \Delta\vec{\mathbf{p}} \qquad \text{9–6}$$

Magnitude of the Impulse and Force
Since an impulse is often delivered in a very short time interval, the average force can be large.

9–4 CONSERVATION OF LINEAR MOMENTUM

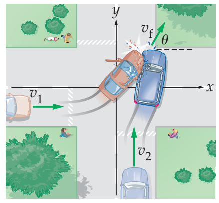


The momentum of an object is conserved (remains constant) if the net force acting on it is zero.

Internal/External Forces
In a system of objects, internal forces always sum to zero. The net force acting on a system of objects, then, is the sum of the external forces.

Conservation of Momentum in a System
In a system of objects, the net momentum is conserved if the net external force acting on the system is zero.

9–5 INELASTIC COLLISIONS



In collisions, we assume that external forces either sum to zero or are small enough to be ignored. Hence, momentum is conserved in all collisions.

Inelastic Collision
In an inelastic collision, the final kinetic energy is different from the initial kinetic energy. The kinetic energy is usually less after a collision, but it can also be more than the initial kinetic energy.

Completely Inelastic Collision
A collision in which objects hit and stick together is referred to as completely inelastic.

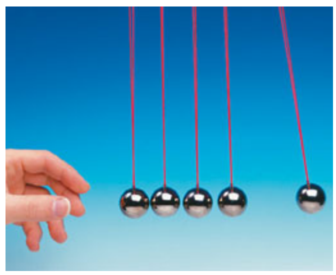
Collisions in One Dimension
A one-dimensional collision occurs along a line, which we can choose to be the x axis. After the collision, the x component of momentum is equal to the x component of momentum before the collision; that is, the x component of momentum is conserved.

If two objects, of mass m_1 and m_2 and with initial velocities $v_{1,i}$ and $v_{2,i}$, collide and stick, the final velocity is

$$v_f = \frac{m_1 v_{1,i} + m_2 v_{2,i}}{m_1 + m_2} \qquad \text{9–10}$$

Collisions in Two Dimensions
In a two-dimensional collision, there are two separate momentum relations to be satisfied: (i) the x component of momentum is conserved, and (ii) the y component of momentum is conserved.

9–6 ELASTIC COLLISIONS



In collisions, we assume that external forces either sum to zero or are small enough to be ignored. Hence, momentum is conserved in all collisions.

Elastic Collision
In an elastic collision, the final kinetic energy is equal to the initial kinetic energy.

Collisions in One Dimension
In an elastic collision in one dimension where mass m_1 is moving with an initial velocity v_0, and mass m_2 is initially at rest, the velocities of the masses after the collision are:

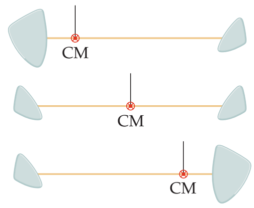
$$v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_0$$

and

$$v_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_0 \qquad \text{9–12}$$

Collisions in Two Dimensions
In elastic collisions in two dimensions, three separate conditions are satisfied: (i) kinetic energy is conserved, (ii) the x component of momentum is conserved, and (iii) the y component of momentum is conserved.

9–7 CENTER OF MASS



The location of the center of mass of a two-dimensional system of objects is defined as follows:

$$X_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum mx}{M} \qquad 9\text{–}14$$

and

$$Y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum my}{M} \qquad 9\text{–}15$$

Motion of the Center of Mass
The velocity of the center of mass is

$$\vec{\mathbf{V}}_{cm} = \frac{m_1 \vec{\mathbf{v}}_1 + m_2 \vec{\mathbf{v}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m\vec{\mathbf{v}}}{M} \qquad 9\text{–}16$$

Note that $M\vec{\mathbf{V}}_{cm} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots = \vec{\mathbf{p}}_{total}$. If a system's momentum is conserved, its center of mass has constant velocity. Similarly, the acceleration of the center of mass is

$$\vec{\mathbf{A}}_{cm} = \frac{m_1 \vec{\mathbf{a}}_1 + m_2 \vec{\mathbf{a}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m\vec{\mathbf{a}}}{M} \qquad 9\text{–}17$$

Note that $M\vec{\mathbf{A}}_{cm} = m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots =$ (net external force). That is,

$$M\vec{\mathbf{A}}_{cm} = \vec{\mathbf{F}}_{net,ext} \qquad 9\text{–}18$$

The center of mass accelerates as if the net external force acted on a single object of mass $M = m_1 + m_2 + \cdots$.

*9–8 SYSTEMS WITH CHANGING MASS: ROCKET PROPULSION

The mass of a rocket changes because its engines expel fuel when they are fired. If fuel is expelled with the speed v and at the rate $\Delta m/\Delta t$, the thrust experienced by the rocket is

$$\text{thrust} = \left(\frac{\Delta m}{\Delta t}\right) v \qquad 9\text{–}19$$

PROBLEM-SOLVING SUMMARY

Type of Calculation	Relevant Physical Concepts	Related Examples
Calculate the momentum of a system.	Each object in a system has a momentum of magnitude mv that points in the direction of its velocity vector. The total momentum is the vector sum of the individual momenta.	Example 9–1
Relate force and time to the impulse.	The impulse acting on a system is the average force, F_{av}, times the time interval, Δt.	Example 9–2 Active Example 9–1
Apply momentum conservation.	Momentum is conserved when the net external force acting on a system is zero.	Examples 9–3, 9–4, 9–5, 9–6, 9–7 Active Example 9–2
Find the center of mass.	The location of the center of mass is given by Equations 9–14 and 9–15.	Example 9–8
Determine the motion of the center of mass.	The center of mass moves the same as if it were a point particle of mass M (the total mass of the system) acted on by the net external force, $\vec{\mathbf{F}}_{net,ext}$.	Example 9–9 Active Example 9–3

CONCEPTUAL QUESTIONS

For instructor-assigned homework, go to www.masteringphysics.com

(Answers to odd-numbered Conceptual Questions can be found in the back of the book.)

1. If you drop your keys, their momentum increases as they fall. Why is the momentum of the keys not conserved? Does this mean that the momentum of the universe increases as the keys fall? Explain.
2. By what factor does an object's kinetic energy change if its speed is doubled? By what factor does its momentum change?
3. A system of particles is known to have zero kinetic energy. What can you say about the momentum of the system?
4. A system of particles is known to have zero momentum. Does it follow that the kinetic energy of the system is also zero? Explain.
5. On a calm day you connect an electric fan to a battery on your sailboat and generate a breeze. Can the wind produced by the fan be used to power the sailboat? Explain.
6. In the previous question, can you use the wind generated by the fan to move a boat that has no sail? Explain why or why not.
7. Crash statistics show that it is safer to be riding in a heavy car in an accident than in a light car. Explain in terms of physical principles.
8. **(a)** As you approach a stoplight, you apply the brakes and bring your car to rest. What happened to your car's initial momentum? **(b)** When the light turns green, you accelerate until you reach cruising speed. What force was responsible for increasing your car's momentum?
9. An object at rest on a frictionless surface is struck by a second object. Is it possible for both objects to be at rest after the collision? Explain.
10. In the previous question, is it possible for one of the two objects to be at rest after the collision? Explain.
11. **(a)** Can two objects on a horizontal frictionless surface have a collision in which all the initial kinetic energy of the system is lost? Explain, and give a specific example if your answer is yes. **(b)** Can two such objects have a collision in which all the initial momentum of the system is lost? Explain, and give a specific example if your answer is yes.
12. Two cars collide at an intersection. If the cars do not stick together, can we conclude that their collision was elastic? Explain.
13. At the instant a bullet is fired from a gun, the bullet and the gun have equal and opposite momenta. Which object—the bullet or the gun—has the greater kinetic energy? Explain. How does your answer apply to the observation that it is safe to hold a gun while it is fired, whereas the bullet is deadly?
14. An hourglass is turned over, and the sand is allowed to pour from the upper half of the glass to the lower half. If the hourglass is resting on a scale, and the total mass of the hourglass and sand is M, describe the reading on the scale as the sand runs to the bottom.
15. In the classic movie *The Spirit of St. Louis*, Jimmy Stewart portrays Charles Lindbergh on his history-making transatlantic flight. Lindbergh is concerned about the weight of his fuel-laden airplane. As he flies over Newfoundland he notices a fly on the dashboard. Speaking to the fly, he wonders aloud, "Does the plane weigh less if you fly inside it as it's flying? Now that's an interesting question." What do you think?
16. A tall, slender drinking glass with a thin base is initially empty. **(a)** Where is the center of mass of the glass? **(b)** Suppose the glass is now filled slowly with water until it is completely full. Describe the position and motion of the center of mass during the filling process.
17. Lifting one foot into the air, you balance on the other foot. What can you say about the location of your center of mass?
18. In the "Fosbury flop" method of high jumping, named for the track and field star Dick Fosbury, an athlete's center of mass may pass under the bar while the athlete's body passes over the bar. Explain how this is possible.

The "Fosbury flop." (Conceptual Question 18)

PROBLEMS AND CONCEPTUAL EXERCISES

Note: Answers to odd-numbered Problems and Conceptual Exercises can be found in the back of the book. **IP** *denotes an integrated problem, with both conceptual and numerical parts;* **BIO** *identifies problems of biological or medical interest;* **CE** *indicates a conceptual exercise.* **Predict/Explain** *problems ask for two responses:* **(a)** *your prediction of a physical outcome, and* **(b)** *the best explanation among three provided. On all problems, red bullets (•, ••, •••) are used to indicate the level of difficulty.*

SECTION 9–1 LINEAR MOMENTUM

1. • Referring to Exercise 9–1, what speed must the baseball have if its momentum is to be equal in magnitude to that of the car? Give your result in miles per hour.
2. • Find the total momentum of the birds in Example 9–1 if the goose reverses direction.
3. •• A 26.2-kg dog is running northward at 2.70 m/s, while a 5.30-kg cat is running eastward at 3.04 m/s. Their 74.0-kg owner has the same momentum as the two pets taken together. Find the direction and magnitude of the owner's velocity.
4. •• **IP** Two air-track carts move toward one another on an air track. Cart 1 has a mass of 0.35 kg and a speed of 1.2 m/s. Cart 2 has a mass of 0.61 kg. **(a)** What speed must cart 2 have if the total momentum of the system is to be zero? **(b)** Since the momentum of the system is zero, does it follow that the kinetic energy of the system is also zero? **(c)** Verify your answer to part (b) by calculating the system's kinetic energy.
5. •• A 0.150-kg baseball is dropped from rest. If the magnitude of the baseball's momentum is 0.780 kg · m/s just before it lands on the ground, from what height was it dropped?
6. •• **IP** A 285-g ball falls vertically downward, hitting the floor with a speed of 2.5 m/s and rebounding upward with a speed of 2.0 m/s. **(a)** Find the magnitude of the change in the ball's momentum. **(b)** Find the change in the magnitude of the ball's momentum. **(c)** Which of the two quantities calculated in parts (a) and (b) is more directly related to the net force acting on the ball during its collision with the floor? Explain.

7. ••• Object 1 has a mass m_1 and a velocity $\vec{\mathbf{v}}_1 = (2.80\text{ m/s})\hat{\mathbf{x}}$. Object 2 has a mass m_2 and a velocity $\vec{\mathbf{v}}_2 = (3.10\text{ m/s})\hat{\mathbf{y}}$. The total momentum of these two objects has a magnitude of $17.6\text{ kg}\cdot\text{m/s}$ and points in a direction 66.5° above the positive x axis. Find m_1 and m_2.

SECTION 9–3 IMPULSE

8. • **CE** Your car rolls slowly in a parking lot and bangs into the metal base of a light pole. In terms of safety, is it better for your collision with the light pole to be elastic, inelastic, or is the safety risk the same for either case? Explain.

9. • **CE Predict/Explain** A net force of 200 N acts on a 100-kg boulder, and a force of the same magnitude acts on a 100-g pebble. **(a)** Is the change of the boulder's momentum in one second greater than, less than, or equal to the change of the pebble's momentum in the same time period? **(b)** Choose the *best explanation* from among the following:
   I. The large mass of the boulder gives it the greater momentum.
   II. The force causes a much greater speed in the 100-g pebble, resulting in more momentum.
   III. Equal force means equal change in momentum for a given time.

10. • **CE Predict/Explain** Referring to the previous question, **(a)** is the change in the boulder's speed in one second greater than, less than, or equal to the change in speed of the pebble in the same time period? **(b)** Choose the *best explanation* from among the following:
   I. The large mass of the boulder results in a small acceleration.
   II. The same force results in the same change in speed for a given time.
   III. Once the boulder gets moving it is harder to stop than the pebble.

11. • **CE Predict/Explain** A friend tosses a ball of mass m to you with a speed v. When you catch the ball, you feel a noticeable sting in your hand, due to the force required to stop the ball. **(a)** If you now catch a second ball, with a mass $2m$ and speed $v/2$, is the sting you feel greater than, less than, or equal to the sting you felt when you caught the first ball? The time required to stop the two balls is the same. **(b)** Choose the *best explanation* from among the following:
   I. The second ball has less kinetic energy, since kinetic energy depends on v^2, and hence it produces less sting.
   II. The two balls have the same momentum, and hence they produce the same sting.
   III. The second ball has more mass, and hence it produces the greater sting.

12. • **CE** Force A has a magnitude F and acts for the time Δt, force B has a magnitude $2F$ and acts for the time $\Delta t/3$, force C has a magnitude $5F$ and acts for the time $\Delta t/10$, and force D has a magnitude $10F$ and acts for the time $\Delta t/100$. Rank these forces in order of increasing impulse. Indicate ties where appropriate.

13. • Find the magnitude of the impulse delivered to a soccer ball when a player kicks it with a force of 1250 N. Assume that the player's foot is in contact with the ball for 5.95×10^{-3} s.

14. • In a typical golf swing, the club is in contact with the ball for about 0.0010 s. If the 45-g ball acquires a speed of 67 m/s, estimate the magnitude of the force exerted by the club on the ball.

15. • A 0.50-kg croquet ball is initially at rest on the grass. When the ball is struck by a mallet, the average force exerted on it is 230 N. If the ball's speed after being struck is 3.2 m/s, how long was the mallet in contact with the ball?

16. • When spiking a volleyball, a player changes the velocity of the ball from 4.2 m/s to −24 m/s along a certain direction. If the impulse delivered to the ball by the player is $-9.3\text{ kg}\cdot\text{m/s}$, what is the mass of the volleyball?

17. •• **IP** A 15.0-g marble is dropped from rest onto the floor 1.44 m below. **(a)** If the marble bounces straight upward to a height of 0.640 m, what are the magnitude and direction of the impulse delivered to the marble by the floor? **(b)** If the marble had bounced to a greater height, would the impulse delivered to it have been greater or less than the impulse found in part (a)? Explain.

18. •• To make a bounce pass, a player throws a 0.60-kg basketball toward the floor. The ball hits the floor with a speed of 5.4 m/s at an angle of 65° to the vertical. If the ball rebounds with the same speed and angle, what was the impulse delivered to it by the floor?

19. •• **IP** A 0.14-kg baseball moves toward home plate with a velocity $\vec{\mathbf{v}}_i = (-36\text{ m/s})\hat{\mathbf{x}}$. After striking the bat, the ball moves vertically upward with a velocity $\vec{\mathbf{v}}_f = (18\text{ m/s})\hat{\mathbf{y}}$. **(a)** Find the direction and magnitude of the impulse delivered to the ball by the bat. Assume that the ball and bat are in contact for 1.5 ms. **(b)** How would your answer to part (a) change if the mass of the ball were doubled? **(c)** How would your answer to part (a) change if the mass of the bat were doubled instead?

20. •• A player bounces a 0.43-kg soccer ball off her head, changing the velocity of the ball from $\vec{\mathbf{v}}_i = (8.8\text{ m/s})\hat{\mathbf{x}} + (-2.3\text{ m/s})\hat{\mathbf{y}}$ to $\vec{\mathbf{v}}_f = (5.2\text{ m/s})\hat{\mathbf{x}} + (3.7\text{ m/s})\hat{\mathbf{y}}$. If the ball is in contact with the player's head for 6.7 ms, what are **(a)** the direction and **(b)** the magnitude of the impulse delivered to the ball?

SECTION 9–4 CONSERVATION OF LINEAR MOMENTUM

21. • In a situation similar to Example 9–3, suppose the speeds of the two canoes after they are pushed apart are 0.58 m/s for canoe 1 and 0.42 m/s for canoe 2. If the mass of canoe 1 is 320 kg, what is the mass of canoe 2?

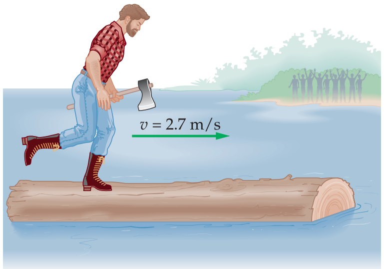
22. • Two ice skaters stand at rest in the center of an ice rink. When they push off against one another the 45-kg skater acquires a speed of 0.62 m/s. If the speed of the other skater is 0.89 m/s, what is this skater's mass?

23. • Suppose the bee in Active Example 9–2 has a mass of 0.175 g. If the bee walks with a speed of 1.41 cm/s relative to the still water, what is the speed of the 4.75-g stick relative to the water?

24. •• An object initially at rest breaks into two pieces as the result of an explosion. One piece has twice the kinetic energy of the other piece. What is the ratio of the masses of the two pieces? Which piece has the larger mass?

25. •• A 92-kg astronaut and a 1200-kg satellite are at rest relative to the space shuttle. The astronaut pushes on the satellite, giving it a speed of 0.14 m/s directly away from the shuttle. Seven and a half seconds later the astronaut comes into contact with the shuttle. What was the initial distance from the shuttle to the astronaut?

26. •• **IP** An 85-kg lumberjack stands at one end of a 380-kg floating log, as shown in **Figure 9–15**. Both the log and the lumberjack are at rest initially. **(a)** If the lumberjack now trots toward the other end of the log with a speed of 2.7 m/s relative to the log, what is the lumberjack's speed relative to the shore? Ignore friction between the log and the water. **(b)** If the mass of the log had been greater, would the lumberjack's speed relative to the shore be greater than, less than, or the same as in part (a)? Explain. **(c)** Check your answer to part (b) by calculating the lumberjack's speed relative to the shore for the case of a 450-kg log.



▲ **FIGURE 9–15** Problem 26

27. ••• A plate drops onto a smooth floor and shatters into three pieces of equal mass. Two of the pieces go off with equal speeds v at right angles to one another. Find the speed and direction of the third piece.

SECTION 9–5 INELASTIC COLLISIONS

28. • A cart of mass m moves with a speed v on a frictionless air track and collides with an identical cart that is stationary. If the two carts stick together after the collision, what is the final kinetic energy of the system?

29. • Suppose the car in Example 9–6 has an initial speed of 20.0 m/s and that the direction of the wreckage after the collision is 40.0° above the x axis. Find the initial speed of the minivan and the final speed of the wreckage.

30. • Two 72.0-kg hockey players skating at 5.45 m/s collide and stick together. If the angle between their initial directions was 115°, what is their speed after the collision?

31. •• **IP** **(a)** Referring to Exercise 9–2, is the final kinetic energy of the car and truck together greater than, less than, or equal to the sum of the initial kinetic energies of the car and truck separately? Explain. **(b)** Verify your answer to part (a) by calculating the initial and final kinetic energies of the system.

32. •• **IP** A bullet with a mass of 4.0 g and a speed of 650 m/s is fired at a block of wood with a mass of 0.095 kg. The block rests on a frictionless surface, and is thin enough that the bullet passes completely through it. Immediately after the bullet exits the block, the speed of the block is 23 m/s. **(a)** What is the speed of the bullet when it exits the block? **(b)** Is the final kinetic energy of this system equal to, less than, or greater than the initial kinetic energy? Explain. **(c)** Verify your answer to part (b) by calculating the initial and final kinetic energies of the system.

33. •• **IP** A 0.420-kg block of wood hangs from the ceiling by a string, and a 0.0750-kg wad of putty is thrown straight upward, striking the bottom of the block with a speed of 5.74 m/s. The wad of putty sticks to the block. **(a)** Is the mechanical energy of this system conserved? **(b)** How high does the putty–block system rise above the original position of the block?

34. •• A 0.430-kg block is attached to a horizontal spring that is at its equilibrium length, and whose force constant is 20.0 N/m. The block rests on a frictionless surface. A 0.0500-kg wad of putty is thrown horizontally at the block, hitting it with a speed of 2.30 m/s and sticking. How far does the putty–block system compress the spring?

35. ••• Two objects moving with a speed v travel in opposite directions in a straight line. The objects stick together when they collide, and move with a speed of $v/4$ after the collision. **(a)** What is the ratio of the final kinetic energy of the system to the initial kinetic energy? **(b)** What is the ratio of the mass of the more massive object to the mass of the less massive object?

SECTION 9–6 ELASTIC COLLISIONS

36. • The collision between a hammer and a nail can be considered to be approximately elastic. Calculate the kinetic energy acquired by a 12-g nail when it is struck by a 550-g hammer moving with an initial speed of 4.5 m/s.

37. • A 732-kg car stopped at an intersection is rear-ended by a 1720-kg truck moving with a speed of 15.5 m/s. If the car was in neutral and its brakes were off, so that the collision is approximately elastic, find the final speed of both vehicles after the collision.

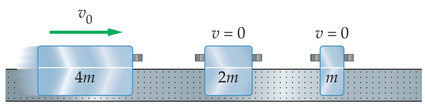
38. • **CE** Suppose you throw a rubber ball at an elephant that is charging directly at you (not a good idea). When the ball bounces back toward you, is its speed greater than, less than, or equal to the speed with which you threw it? Explain.

39. •• **IP** A charging bull elephant with a mass of 5240 kg comes directly toward you with a speed of 4.55 m/s. You toss a 0.150-kg rubber ball at the elephant with a speed of 7.81 m/s. **(a)** When the ball bounces back toward you, what is its speed? **(b)** How do you account for the fact that the ball's kinetic energy has increased?

40. •• **Moderating a Neutron** In a nuclear reactor, neutrons released by nuclear fission must be slowed down before they can trigger additional reactions in other nuclei. To see what sort of material is most effective in slowing (or moderating) a neutron, calculate the ratio of a neutron's final kinetic energy to its initial kinetic energy, K_f/K_i, for a head-on elastic collision with each of the following stationary target particles. (*Note:* The mass of a neutron is $m = 1.009$ u, where the atomic mass unit, u, is defined as follows: $1\ \text{u} = 1.66 \times 10^{-27}$ kg.) **(a)** An electron ($M = 5.49 \times 10^{-4}$ u). **(b)** A proton ($M = 1.007$ u). **(c)** The nucleus of a lead atom ($M = 207.2$ u).

41. •• In the apple-orange collision in Example 9–7, suppose the final velocity of the orange is 1.03 m/s in the negative y direction. What are the final speed and direction of the apple in this case?

42. •• The three air carts shown in **Figure 9–16** have masses, reading from left to right, of $4m$, $2m$, and m, respectively. The most massive cart has an initial speed of v_0; the other two carts are at rest initially. All carts are equipped with spring bumpers that give elastic collisions. **(a)** Find the final speed of each cart. **(b)** Verify that the final kinetic energy of the system is equal to the initial kinetic energy. (Assume the air track is long enough to accommodate all collisions.)



▲ **FIGURE 9–16** Problem 42

43. •• In this problem we show that when one ball is pulled to the left in the photo on page 275, only a single ball recoils to the right—under ideal elastic-collision conditions. To begin, suppose that each ball has a mass m, and that the ball coming in from the left strikes the other balls with a speed v_0. Now, consider the hypothetical case of two balls recoiling to the right. Determine the speed the two recoiling balls must have in order to satisfy **(a)** momentum conservation and **(b)** energy conservation. Since these speeds are not the same, it follows that momentum and energy cannot be conserved simultaneously with a recoil of two balls.

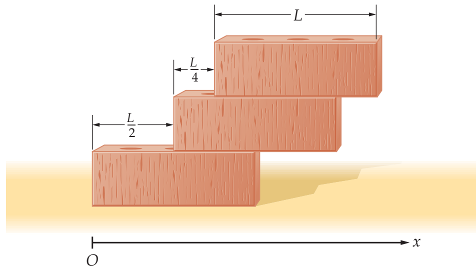
SECTION 9–7 CENTER OF MASS

44. • **CE Predict/Explain** A stalactite in a cave has drops of water falling from it to the cave floor below. The drops are equally

spaced in time and come in rapid succession, so that at any given moment there are many drops in midair. **(a)** Is the center of mass of the midair drops higher than, lower than, or equal to the halfway distance between the tip of the stalactite and the cave floor? **(b)** Choose the *best explanation* from among the following:

I. The drops bunch up as they near the floor of the cave.
II. The drops are equally spaced as they fall, since they are released at equal times.
III. Though equally spaced in time, the drops are closer together higher up.

45. • Find the x coordinate of the center of mass of the bricks shown in **Figure 9–17**.



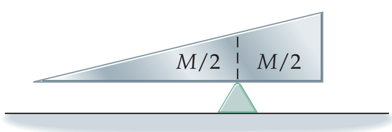
▲ **FIGURE 9–17** Problem 45

46. • You are holding a shopping basket at the grocery store with two 0.56-kg cartons of cereal at the left end of the basket. The basket is 0.71 m long. Where should you place a 1.8-kg half gallon of milk, relative to the left end of the basket, so that the center of mass of your groceries is at the center of the basket?

47. • **Earth–Moon Center of Mass** The Earth has a mass of 5.98×10^{24} kg, the Moon has a mass of 7.35×10^{22} kg, and their center-to-center distance is 3.85×10^{8} m. How far from the center of the Earth is the Earth–Moon center of mass? Is the Earth–Moon center of mass above or below the surface of the Earth? By what distance? (As the Earth and Moon orbit one another, their centers orbit about their common center of mass.)

48. •• **CE Predict/Explain** A piece of sheet metal of mass M is cut into the shape of a right triangle, as shown in **Figure 9–18**. A vertical dashed line is drawn on the sheet at the point where the mass to the left of the line ($M/2$) is equal to the mass to the right of the line (also $M/2$). The sheet is now placed on a fulcrum just under the dashed line and released from rest. **(a)** Does the metal sheet remain level, tip to the left, or tip to the right? **(b)** Choose the *best explanation* from among the following:

I. Equal mass on either side will keep the metal sheet level.
II. The metal sheet extends for a greater distance to the left, which shifts the center of mass to the left of the dashed line.
III. The center of mass is to the right of the dashed line because the metal sheet is thicker there.

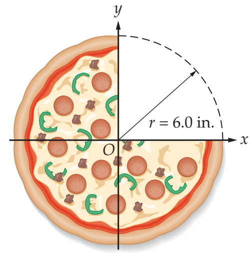


▲ **FIGURE 9–18** Problem 48

49. •• **CE** A pencil standing upright on its eraser end falls over and lands on a table. As the pencil falls, its eraser does not slip. The following questions refer to the contact force exerted on the pencil by the table. Let the positive x direction be in the direction the pencil falls, and the positive y direction be vertically upward. **(a)** During the pencil's fall, is the x component of the contact force positive, negative, or zero? Explain. **(b)** Is the y component of the contact force greater than, less than, or equal to the weight of the pencil? Explain.

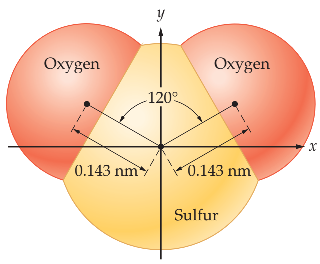
50. •• A cardboard box is in the shape of a cube with each side of length L. If the top of the box is missing, where is the center of mass of the open box? Give your answer relative to the geometric center of the box.

51. •• The location of the center of mass of the partially eaten, 12-inch-diameter pizza shown in **Figure 9–19** is $X_{cm} = -1.4$ in. and $Y_{cm} = -1.4$ in. Assuming each quadrant of the pizza to be the same, find the center of mass of the uneaten pizza above the x axis (that is, the portion of the pizza in the second quadrant).



▲ **FIGURE 9–19** Problem 51

52. •• **The Center of Mass of Sulfur Dioxide** Sulfur dioxide (SO_2) consists of two oxygen atoms (each of mass 16 u, where u is defined in Problem 40) and a single sulfur atom (of mass 32 u). The center-to-center distance between the sulfur atom and either of the oxygen atoms is 0.143 nm, and the angle formed by the three atoms is 120°, as shown in **Figure 9–20**. Find the x and y coordinates of the center of mass of this molecule.



▲ **FIGURE 9–20** Problem 52

53. •• **IP** Three uniform metersticks, each of mass m, are placed on the floor as follows: stick 1 lies along the y axis from $y = 0$ to $y = 1.0$ m, stick 2 lies along the x axis from $x = 0$ to $x = 1.0$ m, stick 3 lies along the x axis from $x = 1.0$ m to $x = 2.0$ m. **(a)** Find the location of the center of mass of the metersticks. **(b)** How would the location of the center of mass be affected if the mass of the metersticks were doubled?

54. •• A 0.726-kg rope 2.00 meters long lies on a floor. You grasp one end of the rope and begin lifting it upward with a constant speed of 0.710 m/s. Find the position and velocity of the rope's center of mass from the time you begin lifting the rope to the time the last piece of rope lifts off the floor. Plot your results. (Assume the rope occupies negligible volume directly below the point where it is being lifted.)

55. •• Repeat the previous problem, this time lowering the rope onto a floor instead of lifting it.

56. •• Consider the system shown in **Figure 9–21**. Assume that after the string breaks the ball falls through the liquid with constant speed. If the mass of the bucket *and* the liquid is 1.20 kg, and the

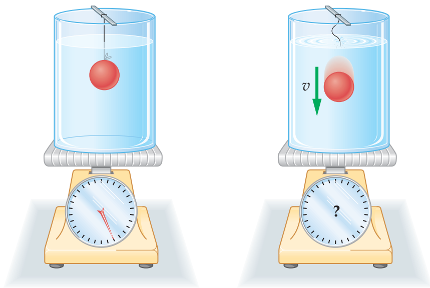


FIGURE 9–21 Problems 56 and 79

mass of the ball is 0.150 kg, what is the reading on the scale **(a)** before and **(b)** after the string breaks?

57. ••• A metal block of mass m is attached to the ceiling by a spring. Connected to the bottom of this block is a string that supports a second block of the same mass m, as shown in **Figure 9–22**. The string connecting the two blocks is now cut. **(a)** What is the net force acting on the two-block system immediately after the string is cut? **(b)** What is the acceleration of the center of mass of the two-block system immediately after the string is cut?

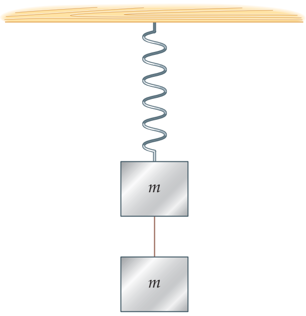


FIGURE 9–22 Problem 57

*SECTION 9–8 SYSTEMS WITH CHANGING MASS: ROCKET PROPULSION

58. • **Helicopter Thrust** During a rescue operation, a 5300-kg helicopter hovers above a fixed point. The helicopter blades send air downward with a speed of 62 m/s. What mass of air must pass through the blades every second to produce enough thrust for the helicopter to hover?

The powerful downdraft from this helicopter's blades creates a circular wave pattern in the water below. The thrust resulting from this downdraft is sufficient to support the weight of the helicopter. (Problem 58)

59. • **Rocks for a Rocket Engine** A child sits in a wagon with a pile of 0.65-kg rocks. If she can throw each rock with a speed of 11 m/s relative to the ground, causing the wagon to move, how many rocks must she throw per minute to maintain a constant average speed against a 3.4-N force of friction?

60. • A 57.8-kg person holding two 0.880-kg bricks stands on a 2.10-kg skateboard. Initially, the skateboard and the person are at rest. The person now throws the two bricks at the same time so that their speed relative to the person is 17.0 m/s. What is the recoil speed of the person and the skateboard relative to the ground, assuming the skateboard moves without friction?

61. •• In the previous problem, calculate the final speed of the person and the skateboard relative to the ground if the person throws the bricks one at a time. Assume that each brick is thrown with a speed of 17.0 m/s relative to the person.

62. •• A 0.540-kg bucket rests on a scale. Into this bucket you pour sand at the constant rate of 56.0 g/s. If the sand lands in the bucket with a speed of 3.20 m/s, **(a)** what is the reading of the scale when there is 0.750 kg of sand in the bucket? **(b)** What is the weight of the bucket and the 0.750 kg of sand?

63. •• **IP** Holding a long rope by its upper end, you lower it onto a scale. The rope has a mass of 0.13 kg per meter of length, and is lowered onto the scale at the constant rate of 1.4 m/s. **(a)** Calculate the thrust exerted by the rope as it lands on the scale. **(b)** At the instant when the amount of rope at rest on the scale has a weight of 2.5 N, does the scale read 2.5 N, more than 2.5 N, or less than 2.5 N? Explain. **(c)** Check your answer to part (b) by calculating the reading on the scale at this time.

GENERAL PROBLEMS

64. • **CE** Object A has a mass m, object B has a mass $2m$, and object C has a mass $m/2$. Rank these objects in order of increasing kinetic energy, given that they all have the same momentum. Indicate ties where appropriate.

65. • **CE** Object A has a mass m, object B has a mass $4m$, and object C has a mass $m/4$. Rank these objects in order of increasing momentum, given that they all have the same kinetic energy. Indicate ties where appropriate.

66. • **CE Predict/Explain** A block of wood is struck by a bullet. **(a)** Is the block more likely to be knocked over if the bullet is metal and embeds itself in the wood, or if the bullet is rubber and bounces off the wood? **(b)** Choose the *best explanation* from among the following:

I. The change in momentum when a bullet rebounds is larger than when it is brought to rest.
II. The metal bullet does more damage to the block.
III. Since the rubber bullet bounces off, it has little effect.

67. • **CE** A juggler performs a series of tricks with three bowling balls while standing on a bathroom scale. Is the average reading of the scale greater than, less than, or equal to the weight of the juggler plus the weight of the three balls? Explain.

68. • A 72.5-kg tourist climbs the stairs to the top of the Washington Monument, which is 555 ft high. How far does the Earth move in the opposite direction as the tourist climbs?

69. •• **CE Predict/Explain** **Figure 9–23** shows a block of mass $2m$ at rest on a horizontal, frictionless table. Attached to this block by a string that passes over a pulley is a second block, with a mass m.

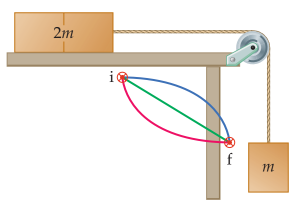


FIGURE 9–23 Problem 69

The initial position of the center of mass of the blocks is indicated by the point i. The blocks are now released and allowed to accelerate; a short time later their center of mass is at the point f. **(a)** Did the center of mass follow the red path, the green path, or the blue path? **(b)** Choose the *best explanation* from among the following:

I. The center of mass must always be closer to the $2m$ block than to the m block.

II. The center of mass starts at rest, and moves in a straight line in the direction of the net force.

III. The masses are accelerating, which implies parabolic motion.

70. •• A car moving with an initial speed v collides with a second stationary car that is one-half as massive. After the collision the first car moves in the same direction as before with a speed $v/3$. **(a)** Find the final speed of the second car. **(b)** Is this collision elastic or inelastic?

71. •• A 1.35-kg block of wood sits at the edge of a table, 0.782 m above the floor. A 0.0105-kg bullet moving horizontally with a speed of 715 m/s embeds itself within the block. What horizontal distance does the block cover before hitting the ground?

72. •• **IP** The carton of eggs shown in **Figure 9–24** is filled with a dozen eggs, each of mass m. Initially, the center of mass of the eggs is at the center of the carton. **(a)** Does the location of the center of mass of the eggs change more if egg 1 is removed or if egg 2 is removed? Explain. **(b)** Find the center of mass of the eggs when egg 1 is removed. **(c)** Find the center of mass of the eggs if egg 2 is removed instead.

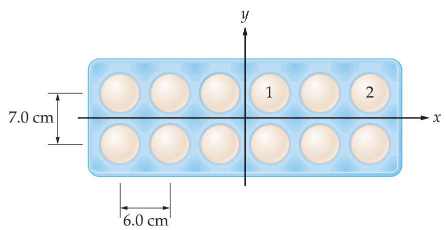


FIGURE 9–24 Problem 72

73. •• **The Force of a Storm** During a severe storm in Palm Beach, FL, on January 2, 1999, 31 inches of rain fell in a period of nine hours. Assuming that the raindrops hit the ground with a speed of 10 m/s, estimate the average upward force exerted by one square meter of ground to stop the falling raindrops during the storm. (*Note:* One cubic meter of water has a mass of 1000 kg.)

74. •• An apple that weighs 2.7 N falls vertically downward from rest for 1.4 s. **(a)** What is the change in the apple's momentum per second? **(b)** What is the total change in its momentum during the 1.4-second fall?

75. •• To balance a 35.5-kg automobile tire and wheel, a mechanic must place a 50.2-g lead weight 25.0 cm from the center of the wheel. When the wheel is balanced, its center of mass is exactly at the center of the wheel. How far from the center of the wheel was its center of mass before the lead weight was added?

76. •• A hoop of mass M and radius R rests on a smooth, level surface. The inside of the hoop has ridges on either side, so that it forms a track on which a ball can roll, as indicated in **Figure 9–25**. If a ball of mass $2M$ and radius $r = R/4$ is released as shown, the system rocks back and forth until it comes to rest with the ball at the bottom of the hoop. When the ball comes to rest, what is the x coordinate of its center?

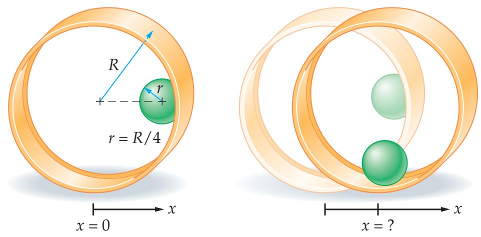


FIGURE 9–25 Problem 76

77. •• **IP** A 63-kg canoeist stands in the middle of her 22-kg canoe. The canoe is 3.0 m long, and the end that is closest to land is 2.5 m from the shore. The canoeist now walks toward the shore until she comes to the end of the canoe. **(a)** When the canoeist stops at the end of her canoe, is her distance from the shore equal to, greater than, or less than 2.5 m? Explain. **(b)** Verify your answer to part (a) by calculating the distance from the canoeist to shore.

78. •• In the previous problem, suppose the canoeist is 3.4 m from shore when she reaches the end of her canoe. What is the canoe's mass?

79. •• Referring to Problem 56, find the reading on the scale **(a)** before and **(b)** after the string breaks, assuming the ball falls through the liquid with an acceleration equal to $0.250g$.

80. •• A young hockey player stands at rest on the ice holding a 1.3-kg helmet. The player tosses the helmet with a speed of 6.5 m/s in a direction 11° above the horizontal, and recoils with a speed of 0.25 m/s. Find the mass of the hockey player.

81. •• Suppose the air carts in Example 9–9 are both moving to the right initially. The cart to the left has a mass m and an initial speed v_0; the cart to the right has an initial speed $v_0/2$. If the center of mass of this system moves to the right with a speed $2v_0/3$, what is the mass of the cart on the right?

82. •• A long, uniform rope with a mass of 0.135 kg per meter lies on the ground. You grab one end of the rope and lift it at the constant rate of 1.13 m/s. Calculate the upward force you must exert at the moment when the top end of the rope is 0.525 m above the ground.

83. •• **The Center of Mass of Water** Find the center of mass of a water molecule, referring to **Figure 9–26** for the relevant angles and distances. The mass of a hydrogen atom is 1.0 u, and the mass of an oxygen atom is 16 u, where u is the atomic mass unit (see Problem 40). Use the center of the oxygen atom as the origin of your coordinate system.

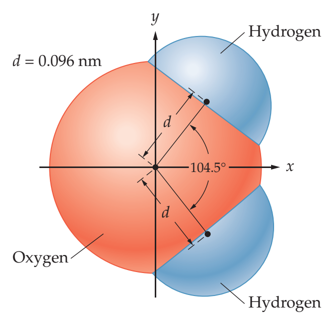
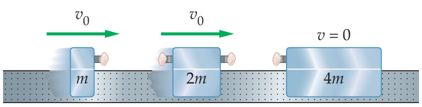


FIGURE 9–26 Problem 83

84. •• The three air carts shown in **Figure 9–27** have masses, reading from left to right, of m, $2m$, and $4m$, respectively. Initially, the cart on the right is at rest, whereas the other two carts are moving to the right with a speed v_0. All carts are equipped with putty bumpers that give completely inelastic collisions. **(a)** Find



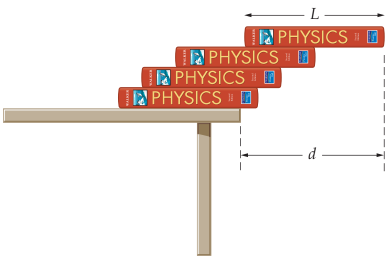
▲ **FIGURE 9–27** Problem 84

the final speed of the carts. **(b)** Calculate the ratio of the final kinetic energy of the system to the initial kinetic energy.

85\. •• **IP** A fireworks rocket is launched vertically into the night sky with an initial speed of 44.2 m/s. The rocket coasts after being launched, then explodes and breaks into two pieces of equal mass 2.50 s later. **(a)** If each piece follows a trajectory that is initially at 45.0° to the vertical, what was their speed immediately after the explosion? **(b)** What is the velocity of the rocket's center of mass before and after the explosion? **(c)** What is the acceleration of the rocket's center of mass before and after the explosion?

86\. •• **IP** The total momentum of two cars approaching an intersection is $\vec{\mathbf{p}}_{\text{total}} = (15{,}000\ \text{kg}\cdot\text{m/s})\hat{\mathbf{x}} + (2100\ \text{kg}\cdot\text{m/s})\hat{\mathbf{y}}$. **(a)** If the momentum of car 1 is $\vec{\mathbf{p}}_1 = (11{,}000\ \text{kg}\cdot\text{m/s})\hat{\mathbf{x}} + (-370\ \text{kg}\cdot\text{m/s})\hat{\mathbf{y}}$, what is the momentum of car 2? **(b)** Does your answer to part (a) depend on which car is closer to the intersection? Explain.

87\. •• **Unlimited Overhang** Four identical textbooks, each of length L, are stacked near the edge of a table, as shown in **Figure 9–28**. The books are stacked in such a way that the distance they overhang the edge of the table, d, is maximized. Find the maximum overhang distance d in terms of L. In particular, show that $d > L$; that is, the top book is completely to the right of the table edge. (In principle, the overhang distance d can be made as large as desired simply by increasing the number of books in the stack.)



▲ **FIGURE 9–28** Problem 87

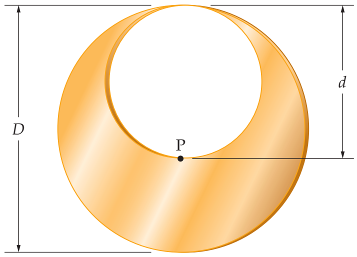
88\. ••• Consider a one-dimensional, head-on elastic collision. One object has a mass m_1 and an initial velocity v_1; the other has a mass m_2 and an initial velocity v_2. Use momentum conservation and energy conservation to show that the final velocities of the two masses are

$$v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)v_2$$

$$v_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2$$

89\. ••• Two air carts of mass $m_1 = 0.84$ kg and $m_2 = 0.42$ kg are placed on a frictionless track. Cart 1 is at rest initially, and has a spring bumper with a force constant of 690 N/m. Cart 2 has a flat metal surface for a bumper, and moves toward the bumper of the stationary cart with an initial speed $v = 0.68$ m/s. **(a)** What is the speed of the two carts at the moment when their speeds are equal? **(b)** How much energy is stored in the spring bumper when the carts have the same speed? **(c)** What is the final speed of the carts after the collision?

90\. ••• **Golden Earrings and the Golden Ratio** A popular earring design features a circular piece of gold of diameter D with a circular cutout of diameter d, as shown in **Figure 9–29**. If this earring is to balance at the point P, show that the diameters must satisfy the condition $D = \phi d$, where $\phi = (1 + \sqrt{5})/2 = 1.61803\ldots$ is the famous "golden ratio."



▲ **FIGURE 9–29** Problem 90

91\. ••• Two objects with masses m_1 and m_2 and initial velocities $v_{1,i}$ and $v_{2,i}$ move along a straight line and collide elastically. Assuming that the objects move along the same straight line after the collision, show that their relative velocities are unchanged; that is, show that $v_{1,i} - v_{2,i} = v_{2,f} - v_{1,f}$. (You can use the results given in Problem 88.)

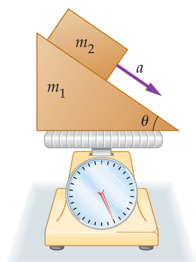
92\. ••• **Amplified Rebound Height** Two small rubber balls are dropped from rest at a height h above a hard floor. When the balls are released, the lighter ball (with mass m) is directly above the heavier ball (with mass M). Assume the heavier ball reaches the floor first and bounces elastically; thus, when the balls collide, the ball of mass M is moving upward with a speed v and the ball of mass m is moving downward with essentially the same speed. In terms of h, find the height to which the ball of mass m rises after the collision. (Use the results given in Problem 88, and assume the balls collide at ground level.)

93\. ••• On a cold winter morning, a child sits on a sled resting on smooth ice. When the 9.75-kg sled is pulled with a horizontal force of 40.0 N, it begins to move with an acceleration of 2.32 m/s^2. The 21.0-kg child accelerates too, but with a smaller acceleration than that of the sled. Thus, the child moves forward relative to the ice, but slides backward relative to the sled. Find the acceleration of the child relative to the ice.

94\. ••• An object of mass m undergoes an elastic collision with an identical object that is at rest. The collision is not head-on. Show that the angle between the velocities of the two objects after the collision is 90°.

95\. ••• **IP Weighing a Block on an Incline** A wedge of mass m_1 is firmly attached to the top of a scale, as shown in **Figure 9–30**. The inclined surface of the wedge makes an angle θ with the horizontal. Now, a block of mass m_2 is placed on the inclined surface of the wedge and allowed to accelerate without friction down the slope. **(a)** Show that the reading on the scale while the block slides is

$$(m_1 + m_2 \cos^2\theta)g$$



▲ **FIGURE 9–30** Problem 95

(b) Explain why the reading on the scale is less than $(m_1 + m_2)g$. **(c)** Show that the expression in part (a) gives the expected results for $\theta = 0$ and $\theta = 90°$.

96. ••• **IP** A uniform rope of length L and mass M rests on a table. **(a)** If you lift one end of the rope upward with a constant *speed*, v, show that the rope's center of mass moves upward with constant *acceleration*. **(b)** Next, suppose you hold the rope suspended in air, with its lower end just touching the table. If you now lower the rope with a constant speed, v, onto the table, is the acceleration of the rope's center of mass upward or downward? Explain your answer. **(c)** Find the magnitude and direction of the acceleration of the rope's center of mass for the case described in part (b). Compare with part (a).

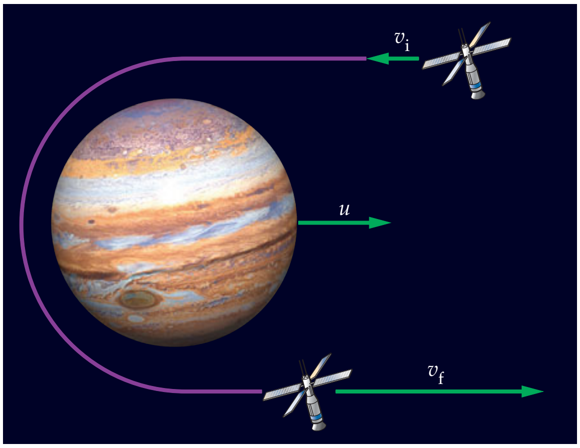
PASSAGE PROBLEMS

Navigating in Space: The Gravitational Slingshot

Many spacecraft navigate through space these days by using the "gravitational slingshot" effect, in which a close encounter with a planet results in a significant increase in magnitude and change in direction of the spacecraft's velocity. In fact, a spacecraft can attain a much greater speed with such a maneuver than it could produce with its own rockets.

The first use of this effect was on February 5, 1974, as the Mariner 10 probe—the first spacecraft to explore Mercury—made a close flyby of the planet Venus on the way to its final destination. More recently, the Cassini probe to Saturn, which was launched on October 15, 1997, and arrived at Saturn on July 1, 2004, made two close passes by Venus, followed by a flyby of Earth and a flyby of Jupiter.

A simplified version of the slingshot maneuver is illustrated in **Figure 9–31**, where we see a spacecraft moving to the left with an initial speed v_i, a planet moving to the right with a speed u, and the same spacecraft moving to the right with a final speed v_f after the encounter. The interaction can be thought of as an elastic collision in one dimension—as if the planet and spacecraft were two air carts on an air track. Both energy and momentum are conserved in this interaction, and hence the following simple condition is satisfied (see Problem 91): The relative speed of approach is equal to the relative speed of departure. This condition, plus the fact that the speed of the massive planet is essentially unchanged, can be used to determine the final speed of the spacecraft.



▲ **FIGURE 9–31** Problems 97, 98, 99, and 100

97. • From the perspective of an observer on the planet, what is the spacecraft's speed of approach?

A. $v_i + u$ **B.** $v_i - u$

C. $u - v_i$ **D.** $v_f - u$

98. • From the perspective of an observer on the planet, what is the spacecraft's speed of departure?

A. $v_f + u$ **B.** $v_f - u$

C. $u - v_f$ **D.** $v_i - u$

99. •• Set the speed of departure from Problem 98 equal to the speed of approach from Problem 97. Solving this relation for the final speed, v_f, yields:

A. $v_f = v_i + u$ **B.** $v_f = v_i - u$

C. $v_f = v_i + 2u$ **D.** $v_f = v_i - 2u$

100. •• Consider the special case in which $v_i = u$. By what factor does the kinetic energy of the spacecraft increase as a result of the encounter?

A. 4 **B.** 8

C. 9 **D.** 16

INTERACTIVE PROBLEMS

101. •• **Referring to Example 9–5** Suppose a bullet of mass $m = 6.75$ g is fired into a ballistic pendulum whose bob has a mass of $M = 0.675$ kg. **(a)** If the bob rises to a height of 0.128 m, what was the initial speed of the bullet? **(b)** What was the speed of the bullet–bob combination immediately after the collision takes place?

102. •• **Referring to Example 9–5** A bullet with a mass $m = 8.10$ g and an initial speed $v_0 = 320$ m/s is fired into a ballistic pendulum. What mass must the bob have if the bullet–bob combination is to rise to a maximum height of 0.125 m after the collision?

103. •• **Referring to Example 9–9** Suppose that cart 1 has a mass of 3.00 kg and an initial speed of 0.250 m/s. Cart 2 has a mass of 1.00 kg and is at rest initially. **(a)** What is the final speed of the carts? **(b)** How much kinetic energy is lost as a result of the collision?

104. •• **Referring to Example 9–9** Suppose the two carts have equal masses and are both moving to the right before the collision. The initial speed of cart 1 (on the left) is v_0 and the initial speed of cart 2 (on the right) is $v_0/2$. **(a)** What is the speed of the center of mass of this system? **(b)** What percentage of the initial kinetic energy is lost as a result of the collision? **(c)** Suppose the collision is elastic. What are the final speeds of the two carts in this case?