11 Rotational Dynamics and Static Equilibrium

Equilibrium, and the sense of serenity that comes with it, involves more than just forces that add to zero—it also depends on where the forces are applied. To keep from falling, for example, the forces exerted on this woman's hand and foot must add up to her total weight. But the total weight must be shared between her hand and foot in just the right way, or else her body will rotate and the pose will be lost. To ensure equilibrium, a new physical quantity—the torque—must also be zero. In this chapter we introduce the torque, and show that equilibrium occurs only when both the net force and the net torque are zero. We will also consider the consequences of nonzero torque.

In the previous chapter we learned how to describe uniformly accelerated rotational motion, but we did not discuss how a given angular acceleration is caused by a given force. The connection between forces and angular acceleration is the focus of this chapter.

We begin by defining a quantity that is the rotational equivalent of force. This quantity is called the *torque*. Although torque may not be as familiar a term as force, your muscles are exerting torques on your body at this very moment. In fact, every time you raise an arm, extend a finger, or stretch a leg, you exert torques to carry out these motions. Thus, our ability to move from place to place, or to hold our body still, is intimately related to our ability to exert precisely controlled torques on our limbs.

We also introduce the notion of *angular momentum* in this chapter and show that it is related to torque in essentially the same way that linear momentum is related to force. As a result, it follows that angular momentum is conserved when the net external torque acting on a system is zero. Thus, conservation of angular momentum joins conservation of energy and conservation of linear momentum as one of the fundamental principles on which all physics is based.

11–1 Torque

Suppose you want to loosen a nut by rotating it counterclockwise with a wrench. If you have ever used a wrench in this way, you probably know that the nut is more likely to turn if you apply your force as far from the nut as possible, as indicated in **Figure 11–1 (a)**. Applying a force near the nut would not be very effective—you could still get the nut to turn, but it would require considerably more effort! Similarly, it is much easier to open a revolving door if you push far from the axis of rotation, as indicated in **Figure 11–1 (b)**. Clearly, then, the tendency for a force to cause a rotation increases with the distance, r, from the axis of rotation to the force. As a result, it is useful to define a quantity called the **torque**, τ, that takes into account both the magnitude of the force, F, *and* the distance from the axis of rotation, r:

The long handle of this wrench enables the user to produce a large torque without having to exert a very great force.

Definition of Torque, τ, for a Tangential Force

$$\tau = rF \qquad 11\text{–}1$$

SI unit: N · m

Note that the torque increases with both the force and the distance.

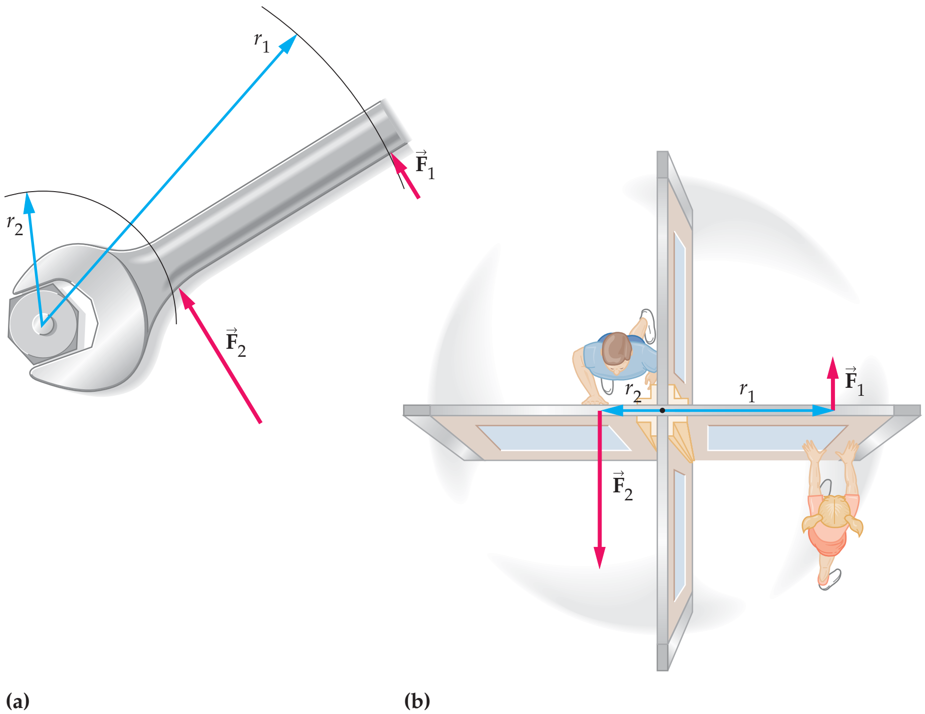


FIGURE 11–1 Applying a torque

(a) When a wrench is used to loosen a nut, less force is required if it is applied far from the nut. **(b)** Similarly, less force is required to open a revolving door if it is applied far from the axis of rotation.

Equation 11–1 is valid only when the applied force is *tangential* to a circle of radius r centered on the axis of rotation, as indicated in Figure 11–1. The more general case is considered later in this section. First, we use Equation 11–1 to determine how much force is needed to open a swinging door, depending on where we apply the force.

EXERCISE 11–1

To open the door in Figure 11–1 (b) a tangential force F is applied at a distance r from the axis of rotation. If the minimum torque required to open the door is 3.1 N · m, what force must be applied if r is **(a)** 0.94 m or **(b)** 0.35 m?

SOLUTION

(a) Setting $\tau = r_1F_1 = 3.1\ \text{N}\cdot\text{m}$, we find that the required force is

$$F_1 = \frac{\tau}{r_1} = \frac{3.1\ \text{N}\cdot\text{m}}{0.94\ \text{m}} = 3.3\ \text{N}$$

PROBLEM-SOLVING NOTE

The Units of Torque

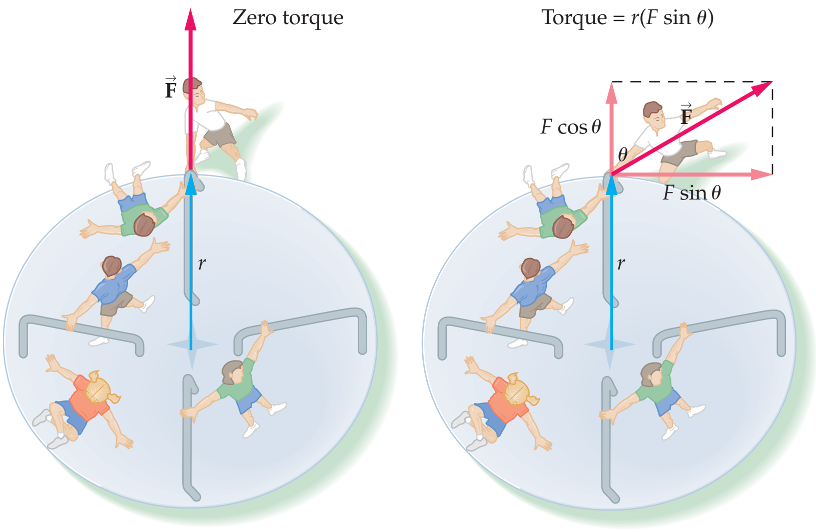
Note that the units of torque are N · m, the same as the units of work. Though their units are the same, torque, τ, and work, W, represent different physical quantities and should not be confused with one another.

(b) Repeat the calculation, this time with $r_2 = 0.35$ m:

$$F_2 = \frac{\tau}{r_2} = \frac{3.1\ \text{N}\cdot\text{m}}{0.35\ \text{m}} = 8.9\ \text{N}$$

As expected, the required force is greater when it is applied closer to the hinges.

To this point we have considered tangential forces only. What happens if you exert a force in a direction that is not tangential? Suppose, for example, that you pull on a playground merry-go-round in a direction that is radial—that is, along a line that extends through the axis of rotation—as in **Figure 11–2 (a)**. In this case, your force has no tendency to cause a rotation. Instead, the axle of the merry-go-round simply exerts an equal and opposite force, and the merry-go-round remains at rest. Similarly, if you were to push or pull in a radial direction on a swinging door it would not rotate. We conclude that a *radial force produces zero torque*.



(a) A radial force produces zero torque

(b) Only the tangential component of force causes a torque

▶ FIGURE 11–2 Only the tangential component of a force causes a torque
(a) A radial force causes no rotation. In this case, the force $\vec{\mathbf{F}}$ is opposed by an equal and opposite force exerted by the axle of the merry-go-round. The merry-go-round does not rotate. **(b)** A force applied at an angle θ with respect to the radial direction. The radial component of this force, $F\cos\theta$, causes no rotation; the tangential component, $F\sin\theta$, can cause a rotation.

On the other hand, what if your force is at an angle θ relative to a radial line, as shown in **Figure 11–2 (b)**? To analyze this case, we first resolve the force vector $\vec{\mathbf{F}}$ into radial and tangential components. Referring to the figure, we see that the radial component has a magnitude of $F\cos\theta$, and the tangential component has a magnitude of $F \sin\theta$. Because it is the tangential component alone that causes rotation, we define the torque to have a magnitude of $r(F\sin\theta)$. That is,

General Definition of Torque, τ

$$\tau = r(F\sin\theta) \qquad 11\text{–}2$$

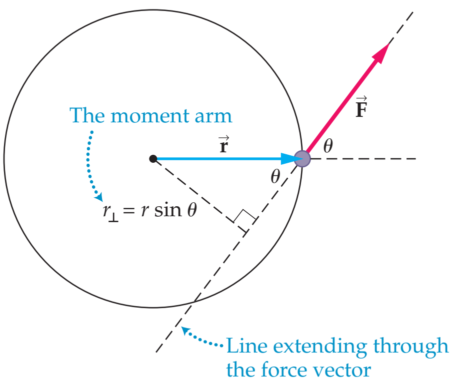
SI units: N · m

(More generally, the torque can be defined as the **cross product** between the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$; that is, $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$. The cross product is discussed in detail in Appendix A.)

As a quick check, note that a radial force corresponds to $\theta = 0$. In this case, $\tau = r(F\sin 0) = 0$, as expected. If the force is tangential, however, it follows that $\theta = \pi/2$. This gives $\tau = r(F\sin\pi/2) = rF$, in agreement with Equation 11–1.

An equivalent way to define the torque is in terms of the **moment arm**, $r_\perp$. The idea here is to extend a line through the force vector, as in **Figure 11–3**, and then draw a second line from the axis of rotation perpendicular to the line of the force. The perpendicular distance from the axis of rotation to the line of the force is defined to be $r_\perp$. From the figure, we see that

$$r_\perp = r\sin\theta$$



▲ FIGURE 11–3 The moment arm
To find the moment arm, $r_\perp$, for a given force, first extend a line through the force vector. This line is sometimes referred to as the "line of action." Next, drop a perpendicular line from the axis of rotation to the line of the force. The perpendicular distance is $r_\perp = r\sin\theta$.

In addition, we note that a simple rearrangement of the torque expression in Equation 11–2 yields

$$\tau = r(F \sin\theta) = (r \sin\theta)F$$

Thus, the torque can be written as the moment arm times the force:

$$\tau = r_{\perp}F \qquad 11\text{–}3$$

Just as a force applied to an object gives rise to a linear acceleration, a torque applied to an object gives rise to an angular acceleration. For example, if a torque acts on an object at rest, the object will begin to rotate; if a torque acts on a rotating object, the object's angular velocity will change. In fact, the greater the torque applied to an object, the greater its angular acceleration, as we shall see in the next section. For this reason, the sign of the torque is determined by the same convention used in Section 10–1 for angular acceleration:

Sign Convention for Torque

By convention, if a torque τ acts alone, then

$\tau > 0$ if the torque causes a counterclockwise angular acceleration
$\tau < 0$ if the torque causes a clockwise angular acceleration

In a system with more than one torque, the sign of each torque is determined by the type of angular acceleration *it alone* would produce. The net torque acting on the system, then, is the sum of each individual torque, taking into account the proper sign. This is illustrated in the following Example.

▲ The net torque on the wheel of this ship is the sum of the torques exerted by the two helmsmen. At the moment pictured, they are both exerting negative torques on the wheel, causing it to rotate in the clockwise direction. This will turn the boat to its left—or, in nautical terms, to port.

PROBLEM-SOLVING NOTE

The Sign of Torques

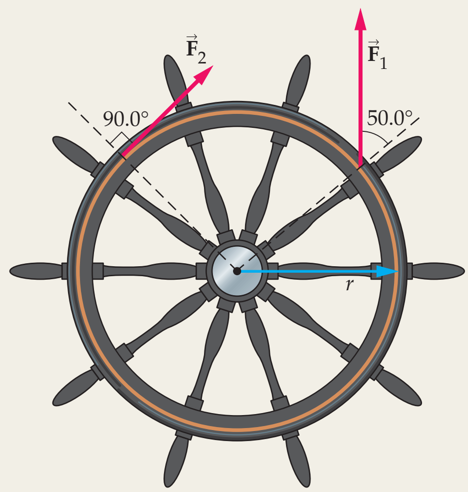
The sign of a torque is determined by the direction of rotation it would cause if it were the only torque acting in the system.

EXAMPLE 11–1 TORQUES TO THE LEFT AND TORQUES TO THE RIGHT

Two helmsmen, in disagreement about which way to turn a ship, exert the forces shown below on a ship's wheel. The wheel has a radius of 0.74 m, and the two forces have the magnitudes $F_1 = 72$ N and $F_2 = 58$ N. Find **(a)** the torque caused by $\vec{\mathbf{F}}_1$ and **(b)** the torque caused by $\vec{\mathbf{F}}_2$. **(c)** In which direction does the wheel turn as a result of these two forces?

PICTURE THE PROBLEM

Our sketch shows that both forces are applied at the distance $r = 0.74$ m from the axis of rotation. However, $F_1 = 72$ N is at an angle of 50.0° relative to the radial direction, whereas $F_2 = 58$ N is tangential, which means that its angle relative to the radial direction is 90.0°.



STRATEGY

For each force, we find the magnitude of the corresponding torque, using $\tau = rF\sin\theta$. As for the signs of the torques, we must consider the angular acceleration each force alone would cause. $\vec{\mathbf{F}}_1$ acting alone would cause the wheel to accelerate counterclockwise, hence its torque is positive. $\vec{\mathbf{F}}_2$ would accelerate the wheel clockwise if it acted alone, hence its torque is negative. If the sum of the two torques is positive, the wheel accelerates counterclockwise; if the sum of the two torques is negative, the wheel accelerates clockwise.

SOLUTION

Part (a)

1. Use Equation 11–2 to calculate the torque due to $\vec{\mathbf{F}}_1$. Recall that this torque is positive: $\tau_1 = rF_1 \sin 50.0° = (0.74\text{ m})(72\text{ N})\sin 50.0° = 41\text{ N}\cdot\text{m}$

Part (b)

2. Similarly, calculate the torque due to $\vec{\mathbf{F}}_2$. Recall that this torque is negative: $\tau_2 = -rF_2 \sin 90.0° = -(0.74\text{ m})(58\text{ N}) = -43\text{ N}\cdot\text{m}$

Part (c)

3. Sum the torques from parts (a) and (b) to find the net torque: $\tau_{net} = \tau_1 + \tau_2 = 41\text{ N}\cdot\text{m} - 43\text{ N}\cdot\text{m} = -2\text{ N}\cdot\text{m}$

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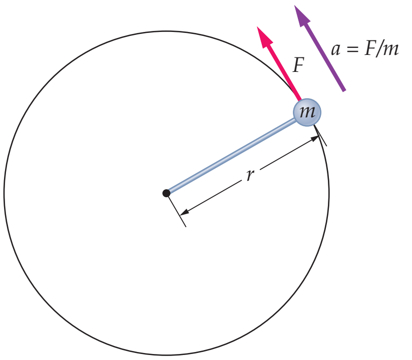
INSIGHT

Because the net torque is negative, the wheel accelerates clockwise. Thus, even though $\vec{\mathbf{F}}_2$ is the smaller force, it has the greater effect in determining the wheel's direction of acceleration. This is because $\vec{\mathbf{F}}_2$ is applied tangentially, whereas $\vec{\mathbf{F}}_1$ is applied in a direction that is partially radial.

PRACTICE PROBLEM

What magnitude of $\vec{\mathbf{F}}_2$ would yield zero net torque on the wheel? [**Answer:** $F_2 = 55$ N]

Some related homework problems: Problem 1, Problem 3



▲ **FIGURE 11–4 Torque and angular acceleration**

A tangential force F applied to a mass m gives it a linear acceleration of magnitude $a = F/m$. The corresponding angular acceleration is $\alpha = \tau/I$, where $\tau = rF$ and $I = mr^2$.

11–2 Torque and Angular Acceleration

In the previous section we indicated that a torque causes a change in the rotational motion of an object. To be more precise, a single torque, τ, acting on an object causes the object to have an angular acceleration, α. In this section we develop the specific relationship between τ and α.

Consider, for example, a small object of mass m connected to an axis of rotation by a light rod of length r, as in **Figure 11–4**. If a tangential force of magnitude F is applied to the mass, it will move with an acceleration given by Newton's second law:

$$a = \frac{F}{m}$$

From Equation 10–14, we know that the linear and angular accelerations are related by

$$\alpha = \frac{a}{r}$$

Combining these results yields the following expression for the angular acceleration:

$$\alpha = \frac{a}{r} = \frac{F}{mr}$$

Finally, multiplying both numerator and denominator by r gives

$$\alpha = \left(\frac{r}{r}\right)\frac{F}{mr} = \frac{rF}{mr^2}$$

Now this last result is rather interesting, since the numerator and denominator have simple interpretations. First, the numerator is the torque, $\tau = rF$, for the case of a tangential force (Equation 11–1). Second, the denominator is the moment of inertia of a single mass m rotating at a radius r; that is, $I = mr^2$. Therefore, we find that

$$\alpha = \frac{rF}{mr^2} = \frac{\tau}{I}$$

or, rewriting slightly,

$$\tau = I\alpha$$

Thus, once we calculate the torque, as described in the previous section, we can find the angular acceleration of a system using $\tau = I\alpha$. Notice that the angular acceleration is directly proportional to the torque, and inversely proportional to the moment of inertia—that is, a large moment of inertia means a small angular acceleration.

Now, the relationship $\tau = I\alpha$ was derived for the special case of a tangential force and a single mass rotating at a radius r. However, the result is completely general. For example, in a system with more than one torque, the relation $\tau = I\alpha$

is replaced with $\tau_{net} = \Sigma\tau = I\alpha$, where τ_{net} is the net torque acting on the system. This gives us the *rotational* version of Newton's second law:

Newton's Second Law for Rotational Motion

$$\sum \tau = I\alpha \qquad 11\text{–}4$$

If only a single torque acts on a system, we will simply write $\tau = I\alpha$.

EXERCISE 11–2

A light rope wrapped around a disk-shaped pulley is pulled tangentially with a force of 0.53 N. Find the angular acceleration of the pulley, given that its mass is 1.3 kg and its radius is 0.11 m.

SOLUTION

The torque applied to the disk is

$$\tau = rF = (0.11\text{ m})(0.53\text{ N}) = 5.8 \times 10^{-2}\text{ N}\cdot\text{m}$$

Since the pulley is a disk, its moment of inertia is given by

$$I = \tfrac{1}{2}mr^2 = \tfrac{1}{2}(1.3\text{ kg})(0.11\text{ m})^2 = 7.9 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Thus, the angular acceleration of the pulley is

$$\alpha = \frac{\tau}{I} = \frac{5.8 \times 10^{-2}\text{ N}\cdot\text{m}}{7.9 \times 10^{-3}\text{ kg}\cdot\text{m}^2} = 7.3\text{ rad/s}^2$$

It is easy to remember the rotational version of Newton's second law, $\Sigma\tau = I\alpha$, by using analogies between rotational and linear quantities. We have already seen that I is the analogue of m, and that α is the analogue of a. Similarly, τ, which causes an angular acceleration, is the analogue of F, which causes a linear acceleration. To summarize:

Linear Quantity	Angular Quantity
m	I
a	α
F	τ

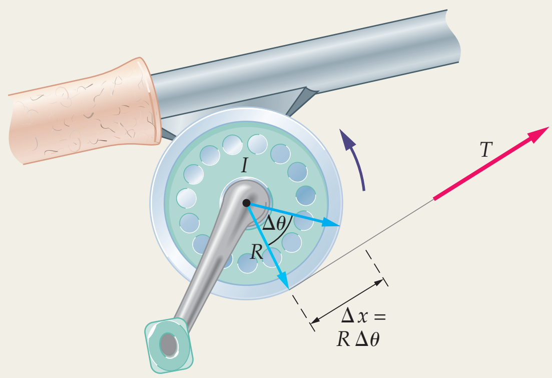
Thus, just as $\Sigma F = ma$ describes linear motion, $\Sigma\tau = I\alpha$ describes rotational motion.

EXAMPLE 11–2 A FISH TAKES THE LINE

A fisherman is dozing when a fish takes the line and pulls it with a tension T. The spool of the fishing reel is at rest initially and rotates without friction (since the fisherman left the drag off) as the fish pulls for a time t. If the radius of the spool is R, and its moment of inertia is I, find **(a)** the angular displacement of the spool, **(b)** the length of line pulled from the spool, and **(c)** the final angular speed of the spool.

PICTURE THE PROBLEM

Our sketch shows the fishing line being pulled tangentially from the spool with a tension T. Because the radius of the spool is R, the torque produced by the line is $\tau = RT$. Also note that as the spool rotates through an angle $\Delta\theta$, the line moves through a linear distance $\Delta x = R\Delta\theta$. Finally, the spool starts at rest, hence $\omega_0 = 0$.



STRATEGY

This is basically an angular kinematics problem, as in Chapter 10, but in this case we must first calculate the angular acceleration using $\alpha = \tau/I$. Once α is known, we can find the angular displacement, $\Delta\theta$, using $\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$. Similarly, we can find the angular speed of the spool, ω, using $\omega = \omega_0 + \alpha t$.

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SOLUTION

1. Calculate the torque acting on the spool. Note that $\theta = 90°$, since the pull is tangential. The radius is $r = R$, and the force applied to the reel is the tension in the line, T:

$$\tau = rF \sin\theta = RT \sin 90° = RT$$

2. Using the result just obtained for the torque, find the angular acceleration of the reel:

$$\alpha = \frac{\tau}{I} = \frac{RT}{I}$$

Part (a)

3. Calculate the angular displacement $\Delta\theta = \theta - \theta_0$:

$$\Delta\theta = \theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\alpha t^2 = \left(\frac{RT}{2I}\right)t^2$$

Part (b)

4. Calculate the length of line pulled from the spool with $\Delta x = R\,\Delta\theta$:

$$\Delta x = R\,\Delta\theta = \left(\frac{R^2T}{2I}\right)t^2$$

Part (c)

5. Use $\omega = \omega_0 + \alpha t$ to find the final angular speed:

$$\omega = \omega_0 + \alpha t = \left(\frac{RT}{I}\right)t$$

INSIGHT

Note that the final angular speed can also be obtained from the kinematic equation relating angular speed and angular distance; $\omega^2 = {\omega_0}^2 + 2\alpha\,\Delta\theta = 0 + 2(RT/I)(RT/2I)t^2 = (RT/I)^2\,t^2$.

This calculation also applies to other situations in which a "line" is pulled from a "reel." Examples include telephone line or sewing thread pulled from a spool.

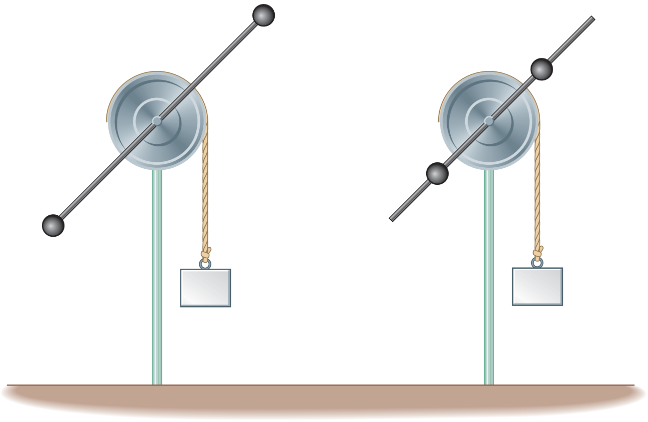
PRACTICE PROBLEM

How fast is the line moving at time t? [**Answer:** $v = R\omega = (R^2T/I)t$]

Some related homework problems: Problem 10, Problem 19

CONCEPTUAL CHECKPOINT 11–1 WHICH BLOCK LANDS FIRST?

The rotating systems shown below differ only in that the two spherical movable masses are positioned either far from the axis of rotation (left), or near the axis of rotation (right). If the hanging blocks are released simultaneously from rest, is it observed that **(a)** the block at left lands first, **(b)** the block at right lands first, or **(c)** both blocks land at the same time?



REASONING AND DISCUSSION

The net external torque, supplied by the hanging blocks, is the same for each of these systems. However, the moment of inertia of the system at right is less than that of the system at left because the movable masses are closer to the axis of rotation. Since the angular acceleration is inversely proportional to the moment of inertia ($\alpha = \tau_{net}/I$), the system at right has the greater angular acceleration, and it wins the race.

ANSWER

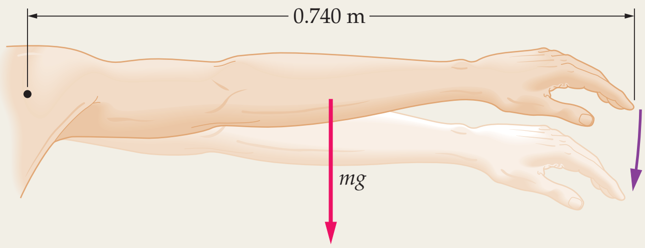
(b) The block at right lands first.

EXAMPLE 11–3 DROP IT

A person holds his outstretched arm at rest in a horizontal position. The mass of the arm is m and its length is 0.740 m. When the person releases his arm, allowing it to drop freely, it begins to rotate about the shoulder joint. Find **(a)** the initial angular acceleration of the arm, and **(b)** the initial linear acceleration of the man's hand. (*Hint:* In calculating the torque, assume the mass of the arm is concentrated at its midpoint. In calculating the angular acceleration, use the moment of inertia of a uniform rod of length L about one end; $I = \frac{1}{3}mL^2$.)

PICTURE THE PROBLEM

The arm is initially horizontal and at rest. When released, it rotates downward about the shoulder joint. The force of gravity, mg, acts at a distance of $(0.740\text{ m})/2 = 0.370\text{ m}$ from the shoulder.



STRATEGY

The angular acceleration, α, can be found using $\tau = I\alpha$. In this case, the initial torque is $\tau = mg(L/2)$, where $L = 0.740$ m, and the moment of inertia is $I = \frac{1}{3}mL^2$.

Once the initial angular acceleration is found, the corresponding linear acceleration is obtained from $a = r\alpha$.

SOLUTION

Part (a)

1. Use $\tau = I\alpha$ to find the angular acceleration, α:

$$\alpha = \frac{\tau}{I}$$

2. Write expressions for the initial torque, τ, and the moment of inertia, I:

$$\tau = mg\frac{L}{2}$$
$$I = \frac{1}{3}mL^2$$

3. Substitute τ and I into the expression for the angular acceleration. Note that the mass of the arm cancels:

$$\alpha = \frac{\tau}{I} = \frac{mgL/2}{mL^2/3} = \frac{3g}{2L}$$

4. Substitute numerical values:

$$\alpha = \frac{3g}{2L} = \frac{3(9.81\text{ m/s}^2)}{2(0.740\text{ m})} = 19.9\text{ rad/s}^2$$

Part (b)

5. Use $a = r\alpha$ to calculate the linear acceleration at the man's hand, a distance $r = L$ from the shoulder:

$$a = L\alpha = L\left(\frac{3g}{2L}\right) = \frac{3}{2}g = 14.7\text{ m/s}^2$$

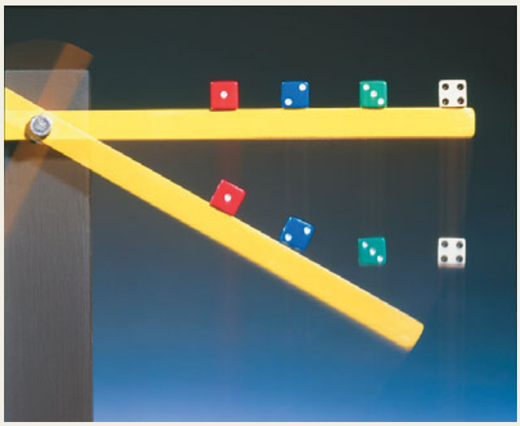
INSIGHT

Note that the linear acceleration of the hand is 1.50 times greater than the acceleration of gravity, regardless of the mass of the arm. This can be demonstrated with the following simple experiment: Hold your arm straight out with a pen resting on your hand. Now, relax your deltoid muscles, and let your arm rotate freely downward about your shoulder joint. Notice that as your arm falls downward, your hand moves more rapidly than the pen, which appears to "lift off" your hand. The pen drops with the acceleration of gravity, which is clearly less than the acceleration of the hand. This effect can be seen in the adjacent photo.

PRACTICE PROBLEM

At what distance from the shoulder is the initial linear acceleration of the arm equal to the acceleration of gravity?
[**Answer:** Set $a = r\alpha$ equal to g. This gives $r = 2L/3 = 0.493$ m.]

Some related homework problems: Problem 13, Problem 15



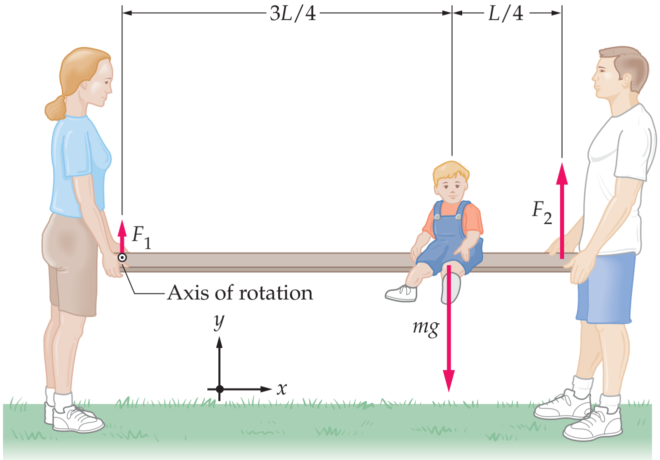
▲ As a rod of length L rotates freely about one end, points farther from the axle than $2L/3$ have an acceleration greater than g (see the Practice Problem for Example 11–3). Thus, the rod falls out from under the last two dice.

11–3 Zero Torque and Static Equilibrium

The parents of a young boy are supporting him on a long, lightweight plank, as illustrated in **Figure 11–5.** If the mass of the child is m, the upward forces exerted by the parents must sum to mg; that is,

$$F_1 + F_2 = mg$$

This condition ensures that the net force acting on the plank is zero. It *does not*, however, guarantee that the plank remains at rest.



▶ FIGURE 11–5 Forces required for static equilibrium

Two parents support a child on a lightweight plank of length L. For the calculation described in the text, we choose the axis of rotation to be the left end of the plank.

To see why, imagine for a moment that the parent on the right lets go of the plank and that the parent on the left increases her force until it is equal to the weight of the child. In this case, $F_1 = mg$ and $F_2 = 0$, which clearly satisfies the force equation we have just written. Since the right end of the plank is no longer supported, however, it drops toward the ground while the left end rises. In other words, the plank rotates in a clockwise sense. For the plank to remain completely at rest, with no translation or rotation, we must impose the following *two* conditions: First, the net force acting on the plank must be zero, so that there is no translational acceleration. Second, the net torque acting on the plank must also be zero, so that there is no rotational acceleration. If both of these conditions are met, an extended object, like the plank, will remain at rest if it starts at rest. To summarize:

Conditions for Static Equilibrium

For an extended object to be in static equilibrium, the following two conditions must be met:

(i) The net force acting on the object must be zero,

$$\sum F_x = 0, \qquad \sum F_y = 0 \qquad \text{11–5}$$

(ii) The net torque acting on the object must be zero,

$$\sum \tau = 0 \qquad \text{11–6}$$

Note that these two conditions are independent of one another; that is, satisfying one does *not* guarantee that the other is satisfied.

Let's apply these conditions to the plank that supports the child. First, we consider the forces acting on the plank, with upward chosen as the positive direction, as in Figure 11–5. Setting the net force equal to zero yields

$$F_1 + F_2 - mg = 0$$

Clearly, this agrees with the force equation we wrote down earlier.

Next, we apply the torque condition. To do so, we must first choose an axis of rotation. For example, we might take the left end of the plank to be the axis, as in Figure 11–5. With this choice, we see that the force F_1 exerts zero torque, since it acts directly through the axis of rotation. On the other hand, F_2 acts at the far end of the plank, a distance L from the axis. In addition, F_2 would cause a counterclockwise (positive) rotation if it acted alone, as we can see in Figure 11–5. Therefore, the torque due to F_2 is

$$\tau_2 = F_2 L$$

Finally, the weight of the child, mg, acts at a distance of $3L/4$ from the axis, and would cause a clockwise (negative) rotation if it acted alone. Hence, its torque is negative:

$$\tau_{mg} = -mg\left(\tfrac{3}{4}L\right)$$

Setting the net torque equal to zero, then, yields the following condition:

$$F_2 L - mg\left(\tfrac{3}{4}L\right) = 0$$

This torque condition, along with the force condition in $F_1 + F_2 - mg = 0$, can be used to determine the two unknowns, F_1 and F_2. For example, we can begin by canceling L in the torque equation to find F_2:

$$F_2 = \tfrac{3}{4}mg$$

Substituting this result into the force condition gives

$$F_1 + \tfrac{3}{4}mg - mg = 0$$

Therefore, F_1 is

$$F_1 = \tfrac{1}{4}mg$$

These two forces support the plank, *and* keep it from rotating. As one might expect, the force nearest the child is greatest.

Our choice of the left end of the plank as the axis of rotation was completely arbitrary. In fact, if an object is in static equilibrium, the net torque acting on it is zero, regardless of the location of the axis of rotation. Hence, we are free to choose an axis of rotation that is most convenient for a given problem. In general, it is useful to pick the axis to be at the location of one of the unknown forces. This eliminates that force from the torque condition, and simplifies the remaining algebra. We consider an alternative choice for the axis of rotation in the following Active Example.

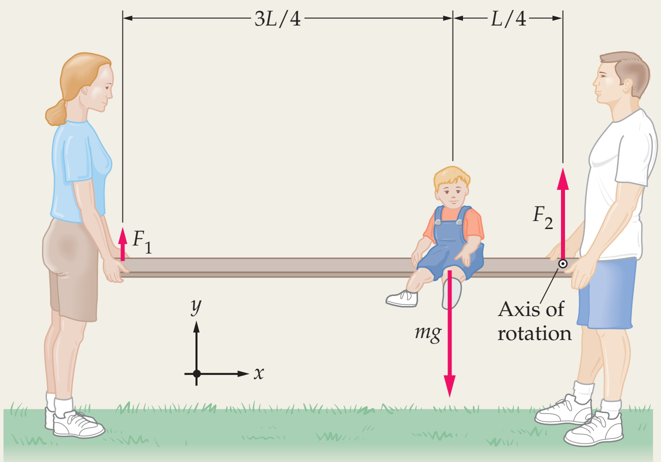
PROBLEM-SOLVING NOTE

Axis of Rotation

Any point in a system may be used as the axis of rotation when calculating torque. It is generally best, however, to choose an axis that gives zero torque for at least one of the unknown forces in the system. Such a choice simplifies the algebra needed to solve for the forces.

ACTIVE EXAMPLE 11–1 FIND THE FORCES: AXIS ON THE RIGHT

A child of mass m is supported on a light plank by his parents, who exert the forces F_1 and F_2 as indicated. Find the forces required to keep the plank in static equilibrium. Use the right end of the plank as the axis of rotation.



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SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Set the net force acting on the plank equal to zero: $F_1 + F_2 - mg = 0$
2. Set the net torque acting on the plank equal to zero: $-F_1(L) + mg\left(\frac{1}{4}L\right) = 0$
3. Note that the torque condition involves only one of the two unknowns, F_1. Use this condition to solve for F_1: $F_1 = \frac{1}{4}mg$
4. Substitute F_1 into the force condition to solve for F_2: $F_2 = mg - \frac{1}{4}mg = \frac{3}{4}mg$

INSIGHT

As expected, the results are identical to those obtained previously. Note that in this case the torque produced by the child would cause a counterclockwise rotation, hence it is positive. Thus, the magnitude *and* sign of the torque produced by a given force depend on the location chosen for the axis of rotation.

YOUR TURN

Suppose the child moves to a new position, with the result that the force exerted by the father is reduced to $0.60mg$. Did the child move to the left or to the right? How far did the child move?

(Answers to **Your Turn** *problems are given in the back of the book.)*

A third choice for the axis of rotation is considered in Problem 24. As expected, all three choices give the same results.

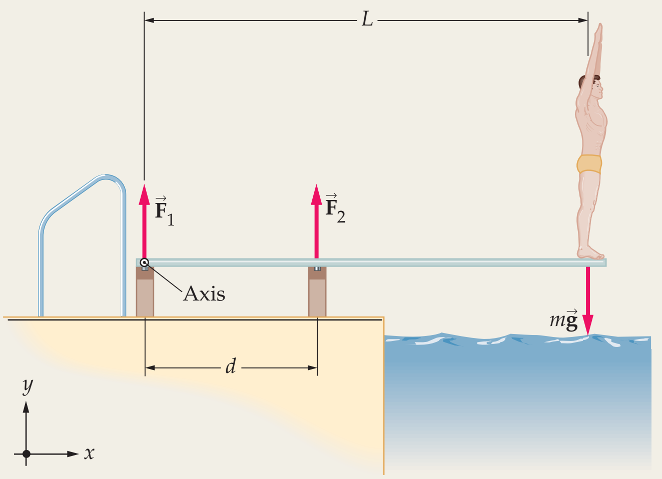
In the next Example, we show that the forces supporting a person or other object sometimes act in different directions. To emphasize the direction of the forces, we solve the Example in terms of the components of the relevant forces.

EXAMPLE 11–4 TAKING THE PLUNGE

A 5.00-m-long diving board of negligible mass is supported by two pillars. One pillar is at the left end of the diving board, as shown below; the other is 1.50 m away. Find the forces exerted by the pillars when a 90.0-kg diver stands at the far end of the board.

PICTURE THE PROBLEM

We choose upward to be the positive direction for the forces. When calculating torques, we use the left end of the diving board as the axis of rotation. Note that $\vec{\mathbf{F}}_2$ would cause a counterclockwise rotation if it acted alone, so its torque is positive. On the other hand, $m\vec{\mathbf{g}}$ would cause a clockwise rotation, so its torque is negative. Finally, $\vec{\mathbf{F}}_2$ acts at a distance d from the axis of rotation, and $m\vec{\mathbf{g}}$ acts at a distance L.



STRATEGY

As usual in static equilibrium problems, we use the conditions of (i) zero net force and (ii) zero net torque to determine the unknown forces, $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$. In this system all forces act in the positive or negative y direction; thus we need only set the net y component of force equal to zero.

SOLUTION

1. Set the net y component of force acting on the diving board equal to zero: $\sum F_y = F_{1,y} + F_{2,y} - mg = 0$
2. Calculate the torque due to each force, using the left end of the board as the axis of rotation. Note that each force is at right angles to the radius and that $\vec{\mathbf{F}}_1$ goes directly through the axis of rotation: $\tau_1 = F_{1,y}(0) = 0$, $\tau_2 = F_{2,y}(d)$, $\tau_3 = -mg(L)$
3. Set the net torque acting on the diving board equal to zero: $\sum \tau = F_{1,y}(0) + F_{2,y}(d) - mg(L) = 0$
4. Solve the torque equation for the force $F_{2,y}$:

$$F_{2,y} = mg(L/d) = (90.0 \text{ kg})(9.81 \text{ m/s}^2)(5.00 \text{ m}/1.50 \text{ m}) = 2940 \text{ N}$$

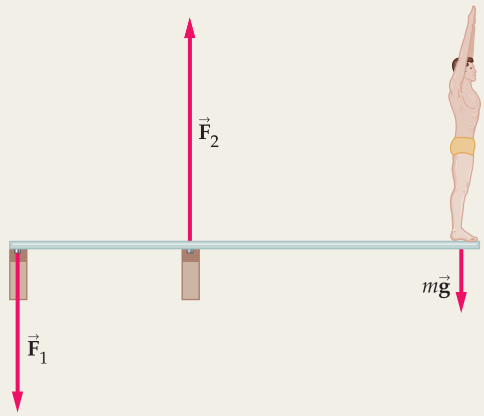
5. Use the force equation to determine $F_{1,y}$:

$$F_{1,y} = mg - F_{2,y}$$
$$= (90.0\ \text{kg})(9.81\ \text{m/s}^2) - 2940\ \text{N} = -2060\ \text{N}$$

INSIGHT

The first point to notice about our solution is that $F_{1,y}$ is negative, which means that $\vec{\mathbf{F}}_1$ is actually directed *downward*, as shown to the right. To see why, imagine for a moment that the board is no longer connected to the first pillar. In this case, the board would rotate clockwise about the second pillar, and the left end of the board would move upward. Thus, a downward force is required on the left end of the board to hold it in place.

The second point is that both pillars exert forces with magnitudes that are considerably larger than the diver's weight, $mg = 883$ N. In particular, the first pillar must pull downward with a force of $2.33mg$, while the second pillar pushes upward with a force of $2.33mg + mg = 3.33mg$. This is not unusual. In fact, it is common for the forces in a structure, such as a bridge, a building, or the human body, to be much greater than the weight it supports.



PRACTICE PROBLEM

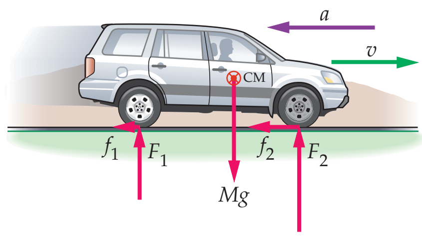
Find the forces exerted by the pillars when the diver is 1.00 m from the right end. [**Answer:** $F_{1,y} = -1470$ N, $F_{2,y} = 2350$ N]

Some related homework problems: Problem 26, Problem 32

To this point we have ignored the mass of the plank holding the child and the diving board holding the swimmer, since they were described as lightweight. If we want to consider the torque exerted by an extended object of finite mass, however, we can simply treat it as if all its mass were concentrated at its center of mass, as was done in similar situations in Section 9–7. We consider such a system in the next Active Example.

REAL-WORLD PHYSICS

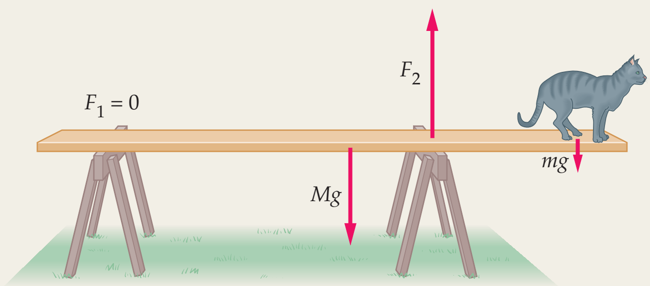
Applying the brakes



▲ As the brakes are applied on this SUV, rotational equilibrium demands that the normal forces exerted on the front tires be greater than the normal forces exerted on the rear tires—which is why braking cars are "nose down" during a rapid stop. For this reason, many cars use disk brakes for the front wheels and less powerful drum brakes for the rear wheels. As the disk brakes wear, they tend to coat the front wheels with dust from the brake pads, which give the front wheels a characteristic "dirty" look.

ACTIVE EXAMPLE 11–2 WALKING THE PLANK: FIND THE MASS

A cat walks along a uniform plank that is 4.00 m long and has a mass of 7.00 kg. The plank is supported by two sawhorses, one 0.440 m from the left end of the board and the other 1.50 m from its right end. When the cat reaches the right end, the plank just begins to tip. What is the mass of the cat?



SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Since the board is just beginning to tip, there is no weight on the left sawhorse: $F_1 = 0$
2. Calculate the torque about the right sawhorse: $Mg(0.500\ \text{m}) - mg(1.50\ \text{m}) = 0$
3. Solve the torque equation for the mass of the cat, m: $m = 0.333M = 2.33$ kg

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INSIGHT

Note that we did not include a torque for the left sawhorse, since F_1 is zero. As an exercise, you might try repeating the calculation with the axis of rotation at the left sawhorse, or at the center of mass of the plank.

YOUR TURN

Write both the zero force and zero torque conditions for the case where the axis of rotation is at the left sawhorse.

(Answers to **Your Turn** *problems are given in the back of the book.)*

Forces with Both Vertical and Horizontal Components

Note that all of the previous examples have dealt with forces that point either directly upward or directly downward. We now consider a more general situation, where forces may have both vertical and horizontal components. For example, consider the wall-mounted lamp (sconce) shown in **Figure 11–6**. The sconce consists of a light curved rod that is bolted to the wall at its lower end. Suspended from the upper end of the rod, a horizontal distance H from the wall, is the lamp of mass m. The rod is also connected to the wall by a horizontal wire a vertical distance V above the bottom of the rod.

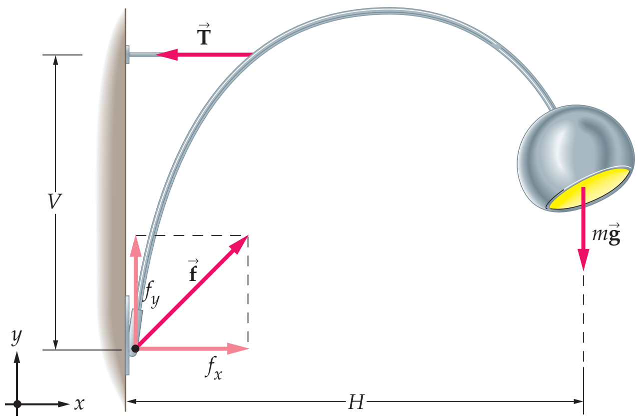


FIGURE 11–6 A lamp in static equilibrium

A wall-mounted lamp of mass m is suspended from a light curved rod. The bottom of the rod is bolted to the wall. The rod is also connected to the wall by a horizontal wire a vertical distance V above the bottom of the rod.

Now, suppose we are designing this sconce to be placed in the lobby of a building on campus. To ensure its structural stability, we would like to know the tension T the wire must exert and the vertical and horizontal components of the force $\vec{f}$ that must be exerted by the bolt on the rod. This information will be important in deciding on the type of wire and bolt to be used in the construction.

To find these forces, we apply the same conditions as before: the net force and the net torque must be zero. In this case, however, forces may have both horizontal and vertical components. Thus, the condition of zero net force is really two separate conditions: (i) zero net force in the horizontal direction; and (ii) zero net force in the vertical direction. These two conditions plus (iii) zero net torque, allow for a full solution of the problem.

We begin with the torque condition. A convenient choice for the axis of rotation is the bottom end of the rod, since this eliminates one of the unknown forces ($\vec{f}$). With this choice we can readily calculate the torques acting on the rod by using the moment arm expression for the torque, $\tau = r_\perp F$ (Equation 11–3). We find

$$\sum \tau = T(V) - mg(H) = 0$$

This relation can be solved immediately for the tension, giving

$$T = mg(H/V)$$

Note that the tension is increased if the wire is connected closer to the bottom of the rod; that is, if V is reduced.

Next, we apply the force conditions. First, we sum the y components of all the forces and set the sum equal to zero:

$$\sum F_y = f_y - mg = 0$$

Thus, the vertical component of the force exerted by the bolt simply supports the weight of the lamp:

$$f_y = mg$$

Finally, we sum the x components of the forces and set that sum equal to zero:

$$\sum F_x = f_x - T = 0$$

Clearly, the x component of the force exerted by the bolt is of the same magnitude as the tension, but it points in the opposite direction:

$$f_x = T = mg(H/V)$$

The bolt, then, pushes upward on the rod to support the lamp, and at the same time it pushes to the right to keep the rod from rotating.

For example, suppose the lamp in Figure 11–6 has a mass of 2.00 kg, and that $V = 12.0$ cm and $H = 15.0$ cm. In this case, we find the following forces:

$$T = mg(H/V) = (2.00\text{ kg})(9.81\text{ m/s}^2)(15.0\text{ cm})/(12.0\text{ cm}) = 24.5\text{ N}$$
$$f_x = T = 24.5\text{ N}$$
$$f_y = mg = (2.00\text{ kg})(9.81\text{ m/s}^2) = 19.6\text{ N}$$

Note that f_x and T are greater than the weight, mg, of the lamp. Just as we found with the diving board in Example 11–4, the forces required of structural elements can be greater than the weight of the object to be supported—an important consideration when designing a structure like a bridge, an airplane, or a sconce. The same effect occurs in the human body. We find in Problem 25, for example, that the force exerted by the biceps to support a baseball in the hand is several times larger than the baseball's weight. Similar conclusions apply to muscles throughout the body.

In Example 11–5 we consider another system in which forces have both vertical and horizontal components.

▲ The chains that support this sign maintain it in a state of translational and rotational equilibrium. The forces in the chains are most easily analyzed by resolving them into vertical and horizontal components and applying the conditions for equilibrium. In particular, the net vertical force, the net horizontal force, and the net torque must all be zero.

REAL-WORLD PHYSICS

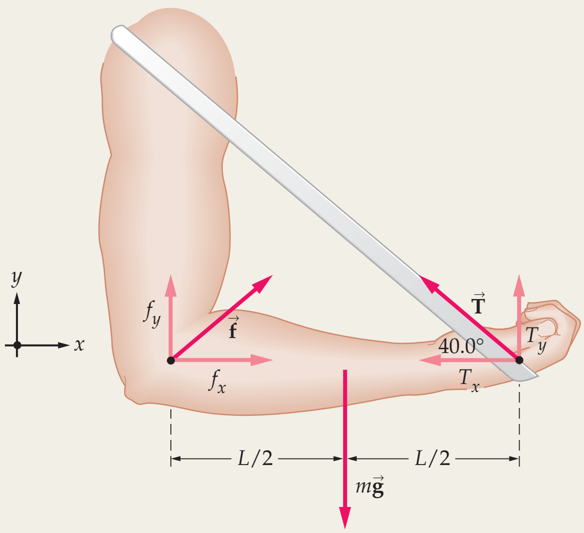
Forces required for structural stability

EXAMPLE 11–5 ARM IN A SLING

A hiker who has broken his forearm rigs a temporary sling using a cord stretching from his shoulder to his hand. The cord holds the forearm level and makes an angle of 40.0° with the horizontal where it attaches to the hand. Considering the forearm and hand to be uniform, with a total mass of 1.30 kg and a length of 0.300 m, find **(a)** the tension in the cord and **(b)** the horizontal and vertical components of the force, $\vec{\mathbf{f}}$, exerted by the humerus (the bone of the upper arm) on the radius and ulna (the bones of the forearm).

PICTURE THE PROBLEM

In our sketch, we use the typical conventions for the positive x and y directions. In addition, since the forearm and hand are assumed to be a uniform object, we indicate the weight mg as acting at its center. The length of the forearm and hand is $L = 0.300$ m. Finally, two other forces act on the forearm: (i) the tension in the cord, $\vec{\mathbf{T}}$, at an angle of 40.0° above the negative x axis, and (ii) the force $\vec{\mathbf{f}}$ exerted at the elbow joint.



STRATEGY

In this system there are three unknowns: T, f_x, and f_y. These unknowns can be determined using the following three conditions: (i) net torque equals zero; (ii) net x component of force equals zero; and (iii) net y component of force equals zero.

We start with the torque condition, using the elbow joint as the axis of rotation. As we shall see, this choice of axis eliminates f, and gives a direct solution for the tension T. Next, we use T and the two force conditions to determine f_x and f_y.

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SOLUTION

Part (a)

1. Calculate the torque about the elbow joint. Note that f causes zero torque, mg causes a negative torque, and the vertical component of T causes a positive torque. The horizontal component of T produces no torque, since it is on a line with the axis:
$$\sum \tau = (T \sin 40.0°)L - mg(L/2) = 0$$

2. Solve the torque condition for the tension, T:
$$T = \frac{mg}{2 \sin 40.0°} = \frac{(1.30 \text{ kg})(9.81 \text{ m/s}^2)}{2 \sin 40.0°} = 9.92 \text{ N}$$

Part (b)

3. Set the sum of the x components of force equal to zero, and solve for f_x:
$$\sum F_x = f_x - T \cos 40.0° = 0$$
$$f_x = T \cos 40.0° = (9.92 \text{ N}) \cos 40.0° = 7.60 \text{ N}$$

4. Set the sum of the y components of force equal to zero, and solve for f_y:
$$\sum F_y = f_y - mg + T \sin 40.0° = 0$$
$$f_y = mg - T \sin 40.0°$$
$$= (1.30 \text{ kg})(9.81 \text{ m/s}^2) - (9.92 \text{ N}) \sin 40.0° = 6.38 \text{ N}$$

INSIGHT

It is not necessary to determine T_x and T_y separately, since we know the direction of the cord. In particular, it is clear from our sketch that the components of $\vec{\mathbf{T}}$ are $T_x = -T \cos 40.0° = -7.60$ N and $T_y = T \sin 40.0° = 6.38$ N.

Did you notice that $\vec{\mathbf{f}}$ is at an angle of 40.0° with respect to the positive x axis, the same angle that $\vec{\mathbf{T}}$ makes with the negative x axis? The reason for this symmetry, of course, is that mg acts at the center of the forearm. If mg were to act closer to the elbow, for example, $\vec{\mathbf{f}}$ would make a larger angle with the horizontal, as we see in the following Practice Problem.

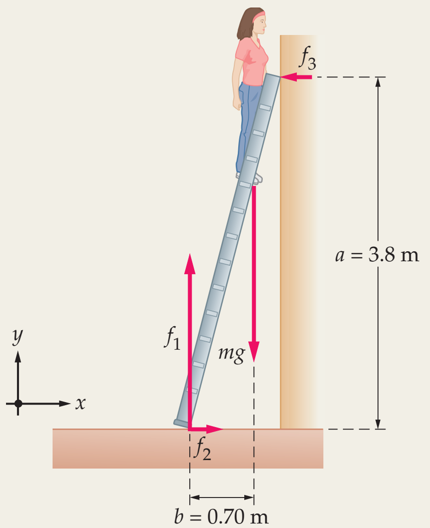
PRACTICE PROBLEM

Suppose the forearm and hand are nonuniform, and that the center of mass is located at a distance of $L/4$ from the elbow joint. What are T, f_x, and f_y in this case? [**Answer:** $T = 4.96$ N, $f_x = 3.80$ N, $f_y = 9.56$ N. In this case, $\vec{\mathbf{f}}$ makes an angle of 68.3° with the horizontal.]

Some related homework problems: Problem 33, Problem 94

ACTIVE EXAMPLE 11–3 DON'T WALK UNDER THE LADDER: FIND THE FORCES

An 85-kg person stands on a lightweight ladder, as shown. The floor is rough; hence, it exerts both a normal force, f_1, and a frictional force, f_2, on the ladder. The wall, on the other hand, is frictionless; it exerts only a normal force, f_3. Using the dimensions given in the figure, find the magnitudes of f_1, f_2, and f_3.



SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Set the net torque acting on the ladder equal to zero. Use the bottom of the ladder as the axis: $f_3(a) - mg(b) = 0$
2. Solve for f_3: $f_3 = mg(b/a) = 150$ N
3. Sum the x components of force and set equal to zero: $f_2 - f_3 = 0$
4. Solve for f_2: $f_2 = f_3 = 150$ N
5. Sum the y components of force and set equal to zero: $f_1 - mg = 0$
6. Solve for f_1: $f_1 = mg = 830$ N

INSIGHT

If the floor is quite smooth, the ladder might slip—it depends on whether the coefficient of static friction is great enough to provide the needed force $f_2 = 150$ N. In this case, the normal force exerted by the floor is $N = f_1 = 830$ N. Therefore, if the coefficient of static friction is greater than 0.18 [since $0.18(830 \text{ N}) = 150$ N], the ladder will stay put. Ladders often have rubberized pads on the bottom in order to increase the static friction, and hence increase the safety of the ladder.

YOUR TURN

Write both the zero force and zero torque conditions for the case where the axis of rotation is at the top of the ladder.

(Answers to **Your Turn** *problems are given in the back of the book.)*

11–4 Center of Mass and Balance

Suppose you decide to construct a mobile. To begin, you tie a thread to a light rod, as in **Figure 11–7**. Note that the rod extends a distance x_1 to the left of the thread and a distance x_2 to the right. At the left end of the rod you attach an object of mass m_1. What mass, m_2, should be attached to the right end if the rod is to be balanced?

From the discussions in the previous sections, it is clear that if the rod is to be in static equilibrium (balanced), the net torque acting on it must be zero. Taking the point where the thread is tied to the rod as the axis of rotation, this zero-torque condition can be written as:

$$m_1 g(x_1) - m_2 g(x_2) = 0$$

Canceling g and rearranging, we find

$$m_1 x_1 = m_2 x_2 \qquad \text{11–7}$$

This gives the following result for m_2:

$$m_2 = m_1(x_1/x_2)$$

For example, if $x_2 = 2x_1$, it follows that m_2 should be one-half of m_1.

Let's now consider a slightly different question: Where is the center of mass of m_1 and m_2? Choosing the origin of the x axis to be at the location of the thread, as indicated in Figure 11–7, we can use the definition of the center of mass, Equation 9–13, to find x_{cm}:

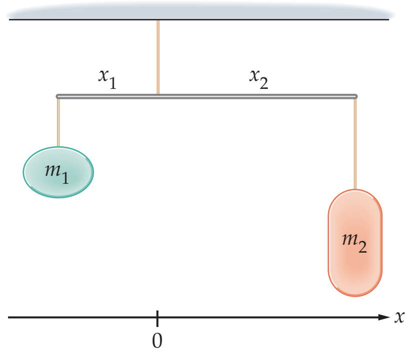
$$x_{cm} = \frac{m_1(-x_1) + m_2(x_2)}{m_1 + m_2} = -\left(\frac{m_1 x_1 - m_2 x_2}{m_1 + m_2}\right)$$

Referring to the zero-torque condition in Equation 11–7, we see that $m_1 x_1 - m_2 x_2 = 0$; hence the center of mass is at the origin:

$$x_{cm} = 0$$

This is precisely where the string is attached. We conclude, then, that the rod balances when the center of mass is directly below the point from which the rod is suspended. This is a general result.

Let's apply this result to the case of the mobile shown in the next Example.



▲ **FIGURE 11–7 Zero torque and balance**
One section of a mobile. The rod is balanced when the net torque acting on it is zero. This is equivalent to having the center of mass directly under the suspension point.

EXAMPLE 11–6 A WELL-BALANCED MEAL

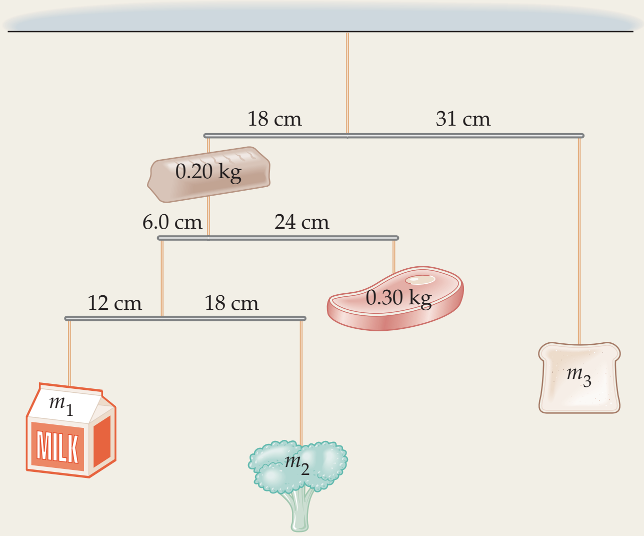
As a grade-school project, students construct a mobile representing some of the major food groups. Their completed artwork is shown below. Find the masses m_1, m_2, and m_3 that are required for a perfectly balanced mobile. Assume the strings and the horizontal rods have negligible mass.

PICTURE THE PROBLEM
The dimensions of the horizontal rods, and the values of the given masses, are indicated in our sketch. Note that each rod is balanced at its suspension point.

STRATEGY
We can find all three unknown masses by repeatedly applying the condition for balance, $m_1 x_1 = m_2 x_2$.

First, we apply the balance condition to m_1 and m_2, with the distances $x_1 = 12$ cm and $x_2 = 18$ cm. This gives a relation between m_1 and m_2.

To get a second relation between m_1 and m_2, we apply the balance condition again at the next higher level of the mobile. That is, the mass $(m_1 + m_2)$ at the distance 6.0 cm must balance the mass 0.30 kg at the distance 24 cm. These two conditions determine m_1 and m_2.



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To find m_3 we again apply the balance condition, this time with the mass $(m_1 + m_2 + 0.30\text{ kg} + 0.20\text{ kg})$ at the distance 18 cm, and the mass m_3 at the distance 31 cm.

SOLUTION

1. Apply the balance condition to m_1 and m_2:
$$m_1(12\text{ cm}) = m_2(18\text{ cm})$$
$$m_1 = (1.5)m_2$$
2. Apply the balance condition to the next level up in the mobile. Solve for the sum, $m_1 + m_2$:
$$(m_1 + m_2)(6.0\text{ cm}) = (0.30\text{ kg})(24\text{ cm})$$
$$m_1 + m_2 = \frac{(0.30\text{ kg})(24\text{ cm})}{6.0\text{ cm}} = 1.2\text{ kg}$$
3. Substitute $m_1 = (1.5)m_2$ into $m_1 + m_2 = 1.2$ kg to find m_2:
$$(1.5)m_2 + m_2 = (2.5)m_2 = 1.2\text{ kg}$$
$$m_2 = 1.2\text{ kg}/2.5 = 0.48\text{ kg}$$
4. Use $m_1 = (1.5)m_2$ to find m_1:
$$m_1 = (1.5)m_2 = (1.5)0.48\text{ kg} = 0.72\text{ kg}$$
5. Apply the balance condition to the top level of the mobile:
$$(0.72\text{ kg} + 0.48\text{ kg} + 0.30\text{ kg} + 0.20\text{ kg})(18\text{ cm}) = m_3(31\text{ cm})$$
6. Solve for m_3:
$$m_3 = \frac{(1.70\text{ kg})(18\text{ cm})}{31\text{ cm}} = 0.99\text{ kg}$$

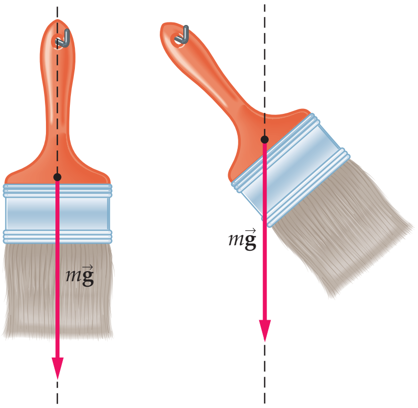
INSIGHT

With the values for m_1, m_2, and m_3 found above, the mobile balances at every level. In fact, the center of mass of the *entire* mobile is directly below the point where the uppermost string attaches to the ceiling.

PRACTICE PROBLEM

Find m_1, m_2, and m_3 if the 0.30-kg mass is replaced with a 0.40-kg mass. [**Answer:** $m_1 = 0.96$ kg, $m_2 = 0.64$ kg, $m_3 = 1.3$ kg]

Some related homework problems: Problem 43, Problem 45



(a) Zero torque **(b)** Nonzero torque

FIGURE 11–8 Equilibrium of a suspended object

(a) If an object's center of mass is directly below the suspension point, its weight creates zero torque and the object is in equilibrium. **(b)** When an object is rotated, so that the center of mass is no longer directly below the suspension point, the object's weight creates a torque. The torque tends to rotate the object to bring the center of mass under the suspension point.

In general, if you allow an arbitrarily shaped object to hang freely, its center of mass is directly below the suspension point. To see why, note that when the center of mass is directly below the suspension point, the torque due to gravity is zero, since the force of gravity extends right through the axis of rotation. This is shown in **Figure 11–8 (a)**. If the object is rotated slightly, as in **Figure 11–8 (b)**, the force of gravity is not in line with the axis of rotation—hence gravity produces a torque. This torque tends to rotate the object, bringing the center of mass back under the suspension point.

For example, suppose you cut a piece of wood into the shape of the continental United States, as shown in **Figure 11–9**, drill a small hole in it, and hang it from the

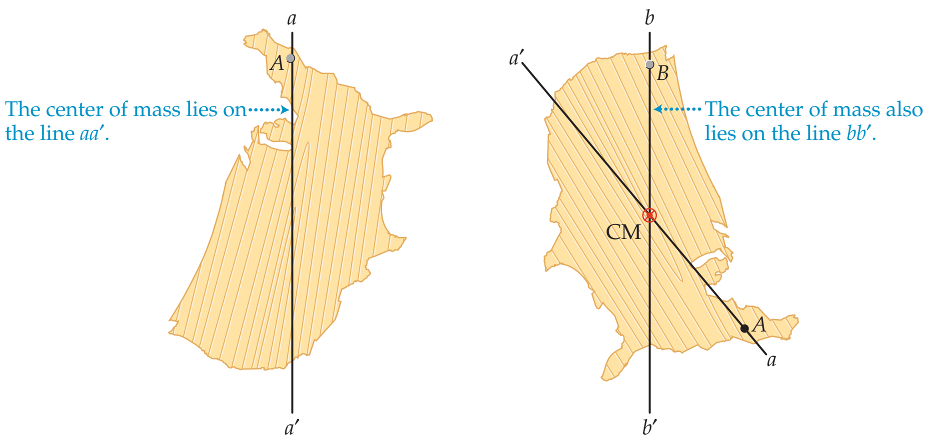


FIGURE 11–9 The geometric center of the United States

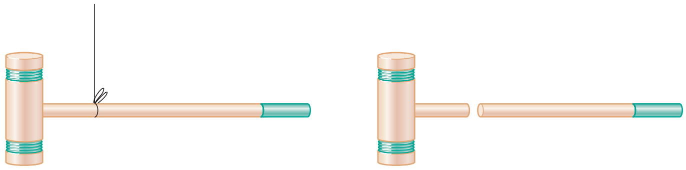
To find the center of mass of an irregularly shaped object, such as a wooden model of the continental United States, suspend it from two or more points. The center of mass lies on a vertical line extending downward from the suspension point. The intersection of these vertical lines gives the precise location of the center of mass.

point A. The result is that the center of mass lies somewhere on the line aa'. Similarly, if a second hole is drilled at point B, we find that the center of mass lies somewhere on the line bb'. The only point that is on both the line aa' *and* the line bb' is the point CM, near Smith Center, Kansas, which marks the location of the center of mass.

▲ In this scene from the movie *Mission Impossible*, Tom Cruise is attempting to download top-secret computer files without setting off the elaborate security system in the room. To accomplish this nearly impossible mission, he is suspended from the ceiling, since touching the floor would immediately give away his presence. To remain in equilibrium above the floor as he works, he must carefully adjust the position of his arms and legs to keep his center of mass directly below the suspension point.

CONCEPTUAL CHECKPOINT 11–2 COMPARE THE MASSES

A croquet mallet balances when suspended from its center of mass, as indicated in the drawing at left. If you cut the mallet in two at its center of mass, as in the drawing at right, how do the masses of the two pieces compare? **(a)** The masses are equal; **(b)** the piece with the head of the mallet has the greater mass; or **(c)** the piece with the head of the mallet has the smaller mass.



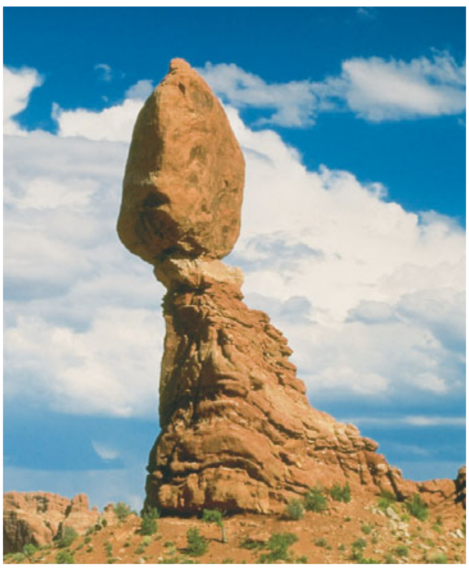
REASONING AND DISCUSSION

The mallet balances because the torques due to the two pieces are of equal magnitude. The piece with the head of the mallet extends a smaller distance from the point of suspension than does the other piece, hence its mass must be greater; that is, a large mass at a small distance creates the same torque as a small mass at a large distance.

ANSWER

(b) The piece with the head of the mallet has the greater mass.

Similar considerations apply to an object that is at rest on a surface, as opposed to being suspended from a point. In such a case, the object is in equilibrium as long as its center of mass is directly above the base on which it is supported. For example, when you stand upright with normal posture your feet provide a base of support, and your center of mass is above a point roughly halfway between your feet. If you lift your right foot from the floor—without changing your posture—you will begin to lose your balance and tip over. The reason is that your center of mass is no longer above the base of support, which is now your left foot. To balance on your left foot, you must lean slightly in that direction so as to position your center of mass directly above the foot. This principle applies to everything from a performer in a high-wire act to one of the "balancing rocks" that are a familiar sight in the desert Southwest. In Problem 44 we apply this condition for stability to a stack of books on the edge of a table.



◀ (Left) Although it looks precarious, this rock in Arches National Park, Utah, has probably been balancing above the desert for many thousands of years. It will remain secure on its perch as long as its center of mass lies above its base of support. (Right) Although her knowledge may be based more on practical experience than on physics, this woman knows exactly what she must do to keep from falling. By extending one leg backward as she leans forward, she keeps her center of mass safely positioned over the foot that supports her.

11–5 Dynamic Applications of Torque

In this section we focus on applications of Newton's second law for rotation. For example, consider a disk-shaped pulley of radius R and mass M with a string wrapped around its circumference, as in **Figure 11–10 (a)**. Hanging from the string is a mass m. When the mass is released, it accelerates downward and the pulley begins to rotate. If the pulley rotates without friction, and the string unwraps without slipping, what are the acceleration of the mass and the tension in the string?

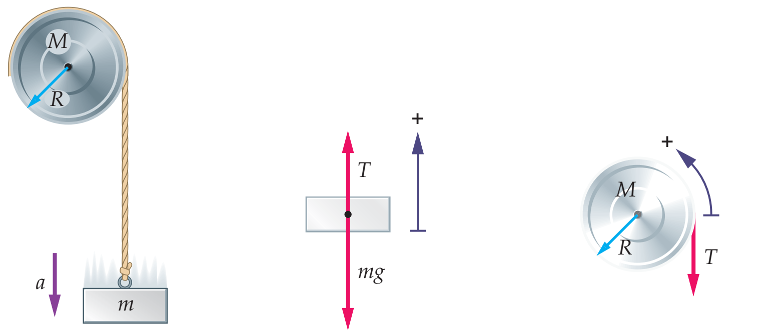


FIGURE 11–10 A mass suspended from a pulley

A mass m hangs from a string wrapped around the circumference of a disk-shaped pulley of radius R and mass M. When the mass is released, it accelerates downward. Positive directions of motion for the system are shown in parts (b) and (c). In part (c), the weight of the pulley acts downward at its center, and the axle exerts an upward force equal in magnitude to the weight of the pulley plus the tension in the string. Of the three forces acting on the pulley, only the tension in the string produces a torque about the axle.

At first it may seem that since the pulley rotates freely, the mass will simply fall with the acceleration of gravity. But remember, the pulley has a nonzero moment of inertia, $I > 0$, which means that it resists any change in its rotational motion. In order for the pulley to rotate, the string must pull downward on it. This means that the string also pulls upward on the mass m with a tension T. As a result, the net downward force on m is less than mg, and thus its acceleration is less than g.

To solve for the acceleration of the mass, we must apply Newton's second law to both the linear motion of the mass *and* the rotational motion of the pulley. The first step is to define a consistent choice of positive directions for the two motions. In Figure 11–10 (a) we note that when the pulley rotates counterclockwise, the mass moves upward. Thus, we choose counterclockwise to be positive for the pulley and upward to be positive for the mass.

With our positive directions established, we proceed to apply Newton's second law. Referring to the free-body diagram for the mass, shown in **Figure 11–10 (b)**, we see that

$$T - mg = ma \tag{11–8}$$

Similarly, the free-body diagram for the pulley is shown in **Figure 11–10 (c)**. Note that the tension in the string, T, exerts a tangential force on the pulley at a distance R from the axis of rotation. This produces a torque of magnitude TR. Since the tension tends to cause a clockwise rotation, it follows that the torque is negative; thus, $\tau = -TR$. As a result, Newton's second law for the pulley gives

$$-TR = I\alpha \tag{11–9}$$

Now, these two statements of Newton's second law are related by the fact that the string unwraps without slipping. As was discussed in Chapter 10, when a string unwraps without slipping, the angular and linear accelerations are related by

$$\alpha = \frac{a}{R}$$

Using this relation in Equation 11–9 we have

$$-TR = I\frac{a}{R}$$

or, dividing by R,

$$T = -I\frac{a}{R^2}$$

Substituting this result into Equation 11–8 yields

$$-I\frac{a}{R^2} - mg = ma$$

Finally, dividing by m and rearranging yields the acceleration, a:

$$a = -\frac{g}{\left(1 + \frac{I}{mR^2}\right)} \qquad \text{11–10}$$

Let's briefly check our solution for a. First, note that a is negative. This is to be expected, since the mass accelerates downward, which is the negative direction. Second, if the moment of inertia were zero, $I = 0$, or if the mass m were infinite, $m \rightarrow \infty$, the mass would fall with the acceleration of gravity, $a = -g$. When I is greater than zero and m is finite, however, the acceleration of the mass has a magnitude less than g. In fact, in the limit of an infinite moment of inertia, $I \rightarrow \infty$, the acceleration vanishes—the mass is simply unable to cause the pulley to rotate in this case.

The next Example presents another system in which Newton's laws are used to relate linear and rotational motions.

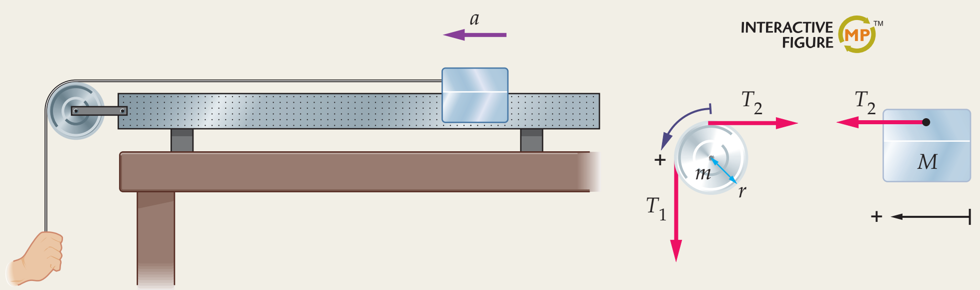
EXAMPLE 11–7 THE PULLEY MATTERS

A 0.31-kg cart on a horizontal air track is attached to a string. The string passes over a disk-shaped pulley of mass 0.080 kg and radius 0.012 m and is pulled vertically downward with a constant force of 1.1 N. Find **(a)** the tension in the string between the pulley and the cart and **(b)** the acceleration of the cart.

PICTURE THE PROBLEM

The system is shown below. We label the mass of the cart with M, the mass of the pulley with m, and the radius of the pulley with r. The applied downward force creates a tension $T_1 = 1.1$ N in the vertical portion of the string. The horizontal portion of the string, from the pulley to the cart, has a tension T_2. If the pulley had zero mass, these two tensions would be equal. In this case, however, T_2 will have a different value than T_1.

We also show the relevant forces acting on the pulley and the cart. The positive direction of rotation is counterclockwise, and the corresponding positive direction of motion for the cart is to the left.



Physical picture Relevant forces

STRATEGY

The two unknowns, T_2 and a, can be found by applying Newton's second law to both the pulley and the cart. This gives two equations for two unknowns.

In applying Newton's second law to the pulley, note that since the pulley is a disk, it follows that $I = \frac{1}{2}mr^2$. Also, since the string is not said to slip as it rotates the pulley, we can assume that the angular and linear accelerations are related by $\alpha = a/r$.

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SOLUTION

Part (a)

1. Apply Newton's second law to the cart: $T_2 = Ma$
2. Apply Newton's second law to the pulley. Note that T_1 causes a positive torque, and T_2 causes a negative torque. In addition, use the relation $\alpha = a/r$: $\sum\tau = I\alpha$

$$rT_1 - rT_2 = \left(\tfrac{1}{2}mr^2\right)\left(\frac{a}{r}\right) = \tfrac{1}{2}mra$$

3. Use the cart equation, $T_2 = Ma$, to eliminate a in the pulley equation:

$$a = \frac{T_2}{M}$$

$$rT_1 - rT_2 = \tfrac{1}{2}mr\left(\frac{T_2}{M}\right)$$

4. Cancel r and solve for T_2:

$$T_2 = \frac{T_1}{1 + m/2M} = \frac{1.1\text{ N}}{1 + 0.080\text{ kg}/[2(0.31\text{ kg})]} = 0.97\text{ N}$$

Part (b)

5. Use $T_2 = Ma$ to find the acceleration:

$$a = \frac{T_2}{M} = \frac{0.97\text{ N}}{0.31\text{ kg}} = 3.1\text{ m/s}^2$$

INSIGHT

Note that T_2 is less than T_1. As a result, the net torque acting on the pulley is in the counterclockwise direction, causing a rotation in that direction, as expected. If the mass of the pulley were zero ($m = 0$), the two tensions would be equal, and the acceleration of the cart would be $T_1/M = 3.5\text{ m/s}^2$.

PRACTICE PROBLEM

What applied force is necessary to give the cart an acceleration of 2.2 m/s^2? [**Answer:** $T_1 = T_2(1 + m/2M) = (Ma)(1 + m/2M) = 0.77\text{ N}$]

Some related homework problems: Problem 49, Problem 50

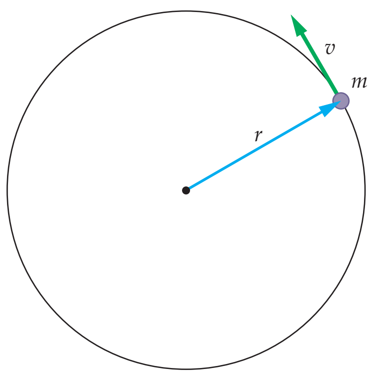


FIGURE 11–11 The angular momentum of circular motion

A particle of mass m, moving in a circle of radius r with a speed v. This particle has an angular momentum of magnitude $L = rmv$.

11–6 Angular Momentum

When an object of mass m moves with a speed v in a straight line, we say that it has a linear momentum, $p = mv$. When the same object moves with an angular speed ω along the circumference of a circle of radius r, as in **Figure 11–11**, we say that it has an **angular momentum**, L. The magnitude of L is given by replacing m and v in the expression for p with their angular analogues I and ω (Section 10–5). Thus, we define the angular momentum as follows:

Definition of the Angular Momentum, *L*

$$L = I\omega \qquad 11\text{–}11$$

SI unit: $\text{kg}\cdot\text{m}^2/\text{s}$

This expression applies to any object undergoing angular motion, whether it is a point mass moving in a circle, as in Figure 11–11, or a rotating hoop, disk, or other object.

Returning for a moment to the case of a point mass m moving in a circle of radius r, recall that the moment of inertia in this case is $I = mr^2$ (Equation 10–18). In addition, the linear speed of the mass is $v = r\omega$ (Equation 10–12). Combining these results, we find

$$L = I\omega = (mr^2)(v/r) = rmv$$

Noting that mv is the linear momentum p, we find that the angular momentum of a point mass can be written in the following form:

$$L = rmv = rp \qquad 11\text{–}12$$

It is important to recall that this expression applies specifically to a point particle moving along the circumference of a circle.

More generally, a point object may be moving at an angle θ with respect to a radial line, as indicated in **Figure 11–12 (a)**. In this case, it is only the tangential component of the momentum, $p \sin\theta = mv \sin\theta$, that contributes to the angular momentum, just as the tangential component of the force, $F \sin\theta$, is all that contributes to the torque. Thus, the magnitude of the angular momentum for a point particle is defined as:

Angular Momentum, *L*, for a Point Particle

$$L = rp \sin\theta = rmv \sin\theta \quad \text{11–13}$$

SI unit: $\text{kg} \cdot \text{m}^2/\text{s}$

Note that if the particle moves in a circular path the angle θ is 90° and the angular momentum is $L = rmv$, in agreement with Equation 11–12. On the other hand, if the object moves radially, so that $\theta = 0$, the angular momentum is zero; $L = rmv \sin 0 = 0$.

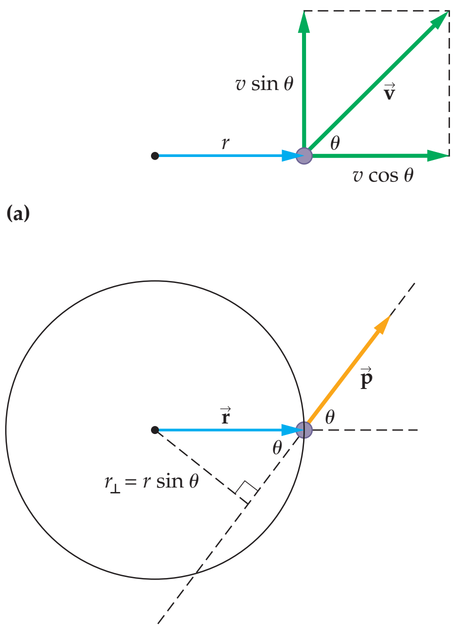


FIGURE 11–12 The angular momentum of nontangential motion

(a) When a particle moves at an angle θ with respect to the radial direction, only the tangential component of velocity, $v \sin\theta$, contributes to the angular momentum. In the case shown here, the particle's angular momentum has a magnitude given by $L = rmv \sin\theta$. **(b)** The angular momentum of an object can also be defined in terms of the moment arm, $r_\perp$. Since $r_\perp = r \sin\theta$, it follows that $L = rmv \sin\theta = r_\perp mv$. Note the similarity between this figure and Figure 11–3.

EXERCISE 11–3

Find the angular momentum of **(a)** a 0.13-kg Frisbee (considered to be a uniform disk of radius 7.5 cm) spinning with an angular speed of 1.15 rad/s, and **(b)** a 95-kg person running with a speed of 5.1 m/s on a circular track of radius 25 m.

SOLUTION

a. Recalling that $I = \frac{1}{2}mR^2$ for a uniform disk (Table 10–1), we have

$$L = I\omega$$
$$= \left(\tfrac{1}{2}mR^2\right)\omega = \tfrac{1}{2}(0.13\ \text{kg})(0.075\ \text{m})^2(1.15\ \text{rad/s}) = 4.2 \times 10^{-4}\ \text{kg} \cdot \text{m}^2/\text{s}$$

b. Treating the person as a particle of mass m, we find

$$L = rmv = (25\ \text{m})(95\ \text{kg})(5.1\ \text{m/s}) = 12{,}000\ \text{kg} \cdot \text{m}^2/\text{s}$$

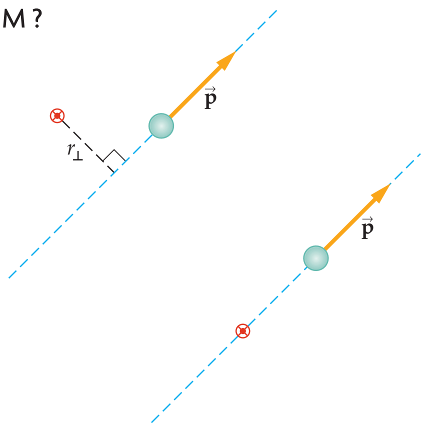
An alternative definition of the angular momentum uses the moment arm, $r_\perp$, as was done for the torque in Equation 11–3. To apply this definition, start by extending a line through the momentum vector, $\vec{p}$, as in **Figure 11–12 (b)**. Next, draw a line from the axis of rotation perpendicular to the line through $\vec{p}$. The perpendicular distance from the axis of rotation to the line of $\vec{p}$ is the moment arm. From the figure we see that $r_\perp = r \sin\theta$. Hence, from Equation 11–13, the angular momentum is

$$L = r_\perp p = r_\perp mv$$

If an object moves in a circle of radius r, the moment arm is $r_\perp = r$ and the angular momentum reduces to our earlier result, $L = rp$.

CONCEPTUAL CHECKPOINT 11–3 ANGULAR MOMENTUM?

Does an object moving in a straight line have nonzero angular momentum **(a)** always, **(b)** sometimes, or **(c)** never?



REASONING AND DISCUSSION

The answer is sometimes, because it depends on the choice of the axis of rotation. If the axis of rotation is not on the line drawn through the momentum vector, as in the left sketch at right, the moment arm is nonzero, and therefore $L = r_\perp p$ is also nonzero. If the axis of rotation is on the line of motion, as in the right sketch, the moment arm is zero; hence the linear momentum is radial and L vanishes.

ANSWER

(b) An object moving in a straight line may or may not have angular momentum, depending on the location of the axis of rotation.

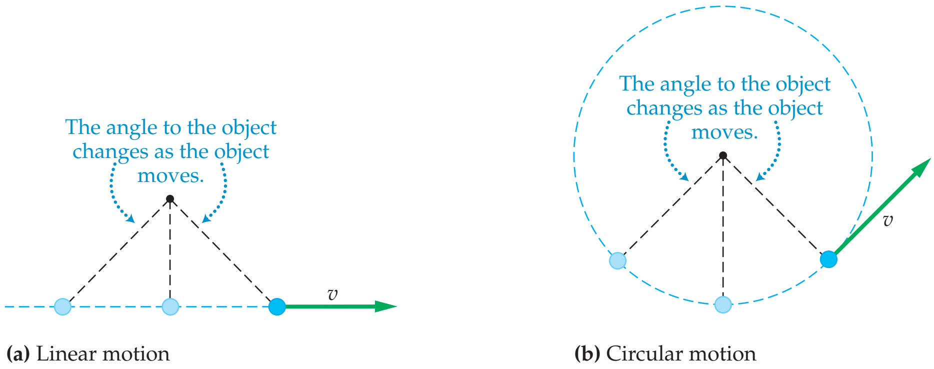


FIGURE 11–13 Angular momentum in linear and circular motion

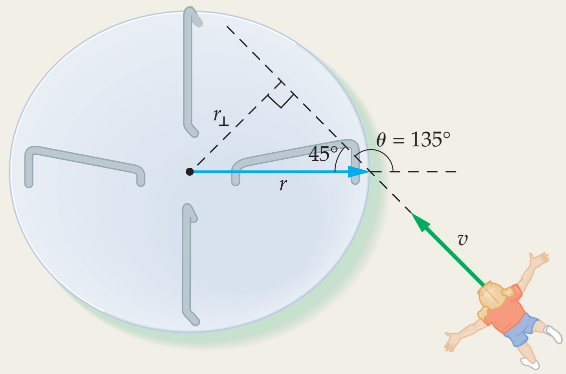
An object moving in **(a)** a straight line and **(b)** a circular path. In both cases, the angular position increases with time; hence, the angular momentum is positive.

Note that an object moving with a momentum p in a straight line that does not go through the axis of rotation has an *angular* position that changes with time. This is illustrated in **Figure 11–13 (a)**. It is for this reason that such an object is said to have an *angular* momentum.

The sign of L is determined by whether the angle to a given object is increasing or decreasing with time. For example, the object moving counterclockwise in a circular path in **Figure 11–13 (b)** has a positive angular momentum, since θ is increasing with time. Similarly, the object in Figure 11–13 (a) also has an angle θ that increases with time, hence its angular momentum is positive as well. On the other hand, if these objects were to have their direction of motion reversed, they would have angles that decrease with time and their angular momenta would be negative.

EXAMPLE 11–8 JUMP ON

Running with a speed of 4.10 m/s, a 21.2-kg child heads toward the rim of a merry-go-round. The radius of the merry-go-round is 2.00 m, and the child moves in the direction indicated. **(a)** What is the child's angular momentum with respect to the center of the merry-go-round? Use $L = rmv \sin\theta$. **(b)** What is the moment arm, $r_\perp$, in this case? **(c)** Find the angular momentum of the child with $L = r_\perp mv$.



PICTURE THE PROBLEM

Our sketch shows the child approaching the rim of the merry-go-round at an angle of 135° relative to the radial direction. Note that the line of motion of the child does not go through the axis of the merry-go-round. As a result, the child has a nonzero angular momentum with respect to that axis of rotation. We also indicate the moment arm, $r_\perp$, and the 45° angle that is opposite to it.

STRATEGY

a. The child's angular momentum can be found by applying $L = rmv \sin\theta$. In this case, we see from the sketch that $\theta = 135°$ and $r = 2.00$ m. The values of m and v are given in the problem statement.

b. and c. Our sketch shows that $r_\perp$ is the side of the right triangle opposite to the angle of 45°. It follows that $r_\perp = r \sin 45°$.

SOLUTION

Part (a)

1. Evaluate $L = rmv \sin\theta$: $L = rmv \sin\theta = (2.00\text{ m})(21.2\text{ kg})(4.10\text{ m/s}) \sin 135°$
$= 123\text{ kg}\cdot\text{m}^2/\text{s}$

Part (b)

2. Calculate the moment arm, $r_\perp$: $r_\perp = r \sin 45° = (2.00\text{ m}) \sin 45° = 1.41\text{ m}$

Part (c)

3. Evaluate $L = r_\perp mv$: $L = r_\perp mv = (1.41\text{ m})(21.2\text{ kg})(4.10\text{ m/s}) = 123\text{ kg}\cdot\text{m}^2/\text{s}$

INSIGHT

When the child lands on the merry-go-round, she will transfer angular momentum to it, causing the merry-go-round to rotate about its center. This will be discussed in more detail in the next section.

Notice that we use 45° in $r_\perp = r \sin 45°$ because we calculate the length of the opposite side of the right triangle indicated in our sketch. We could have used $r_\perp = r \sin 135°$ just as well, using the same angle as in $L = rmv \sin 135°$. The results are the same in either case, since $\sin 135° = \sin 45°$.

PRACTICE PROBLEM

For what angle relative to the radial line does the child have a maximum angular momentum? What is the angular momentum in this case? [**Answer:** $\theta = 90°$, for which $L = rmv = 174\ \text{kg}\cdot\text{m}^2/\text{s}$]

Some related homework problems: Problem 56, Problem 57, Problem 58

Next, we consider the rate of change of angular momentum with time. Since the moment of inertia is a constant—as long as the mass and shape of the object remain unchanged—the change in L in a time interval Δt is

$$\frac{\Delta L}{\Delta t} = I\frac{\Delta\omega}{\Delta t}$$

Recall, however, that $\Delta\omega/\Delta t$ is the angular acceleration, α. Therefore, we have

$$\frac{\Delta L}{\Delta t} = I\alpha$$

Since $I\alpha$ is the torque, it follows that Newton's second law for rotational motion can be written as

Newton's Second Law for Rotational Motion

$$\sum\tau = I\alpha = \frac{\Delta L}{\Delta t} \qquad \text{11–14}$$

Clearly, this is the rotational analogue of $\Sigma F_x = ma_x = \Delta p_x/\Delta t$. Just as force can be expressed as the change in *linear* momentum in a given time interval, the torque can be expressed as the change in *angular* momentum in a time interval.

EXERCISE 11–4

In a light wind, a windmill experiences a constant torque of $255\ \text{N}\cdot\text{m}$. If the windmill is initially at rest, what is its angular momentum 2.00 s later?

SOLUTION

Solve Equation 11–14 for the change in angular momentum due to a single torque τ:

$$\Delta L = L_\text{f} - L_\text{i} = \left(\sum\tau\right)\Delta t = \tau\,\Delta t$$

Since the initial angular momentum of the windmill is zero, its final angular momentum is

$$L_\text{f} = \tau\,\Delta t = (255\ \text{N}\cdot\text{m})(2.00\ \text{s}) = 510\ \text{kg}\cdot\text{m}^2/\text{s}$$

11–7 Conservation of Angular Momentum

When an ice skater goes into a spin and pulls her arms inward to speed up, she probably doesn't think about angular momentum. Neither does a diver who springs into the air and folds her body to speed her rotation. Most people, in fact, are not aware that the actions of these athletes are governed by the same basic laws of physics that cause a collapsing star to spin faster as it becomes a rapidly rotating pulsar. Yet in all these cases, as we shall see, **conservation of angular momentum** is at work.

To see the origin of angular momentum conservation, consider an object with an initial angular momentum L_i acted on by a single torque τ. After a period of time, Δt, the object's angular momentum changes in accordance with Newton's second law:

$$\tau = \frac{\Delta L}{\Delta t}$$

Solving for ΔL, we find

$$\Delta L = L_\text{f} - L_\text{i} = \tau\,\Delta t$$

Thus, the final angular momentum of the object is

$$L_\text{f} = L_\text{i} + \tau\,\Delta t$$

▲ Once she has launched herself into space, this diver is essentially a projectile. However, the principle of conservation of angular momentum allows her to control the rotational part of her motion. By curling her body up into a tight "tuck," she decreases her moment of inertia, thereby increasing the speed of her spin. To slow down for an elegant entry into the water, she will extend her body, increasing her moment of inertia.

If the torque acting on the object is zero, $\tau = 0$, it follows that the initial and final angular momenta are equal—that is, the angular momentum is conserved:

$$L_f = L_i \qquad (\text{if } \tau = 0)$$

Angular momentum is also conserved in systems acted on by more than one torque, provided that the *net external torque* is zero. The reason that internal torques can be ignored is that, just as internal forces come in equal and opposite pairs that cancel, so too do internal torques. As a result, the internal torques in a system sum to zero, and the net torque acting on it is simply the net external torque. Thus, for a general system, angular momentum is conserved if $\tau_{\text{net, ext}}$ is zero:

Conservation of Angular Momentum

$$L_f = L_i \quad (\text{if } \tau_{\text{net, ext}} = 0) \qquad 11\text{–}15$$

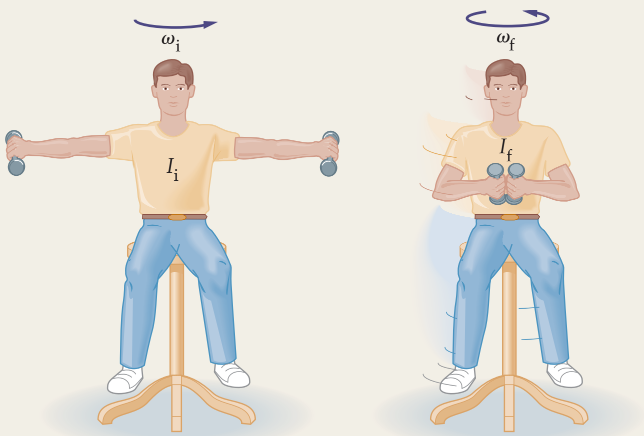
As an illustration of angular momentum conservation, we consider the case of a student rotating on a piano stool in the next Example. Notice how a change in moment of inertia results in a change in angular speed.

EXAMPLE 11–9 GOING FOR A SPIN

For a classroom demonstration, a student sits on a piano stool holding a sizable mass in each hand. Initially, the student holds his arms outstretched and spins about the axis of the stool with an angular speed of 3.72 rad/s. The moment of inertia in this case is $5.33 \text{ kg} \cdot \text{m}^2$. While still spinning, the student pulls his arms in to his chest, reducing the moment of inertia to $1.60 \text{ kg} \cdot \text{m}^2$. **(a)** What is the student's angular speed now? **(b)** Find the initial and final angular momenta of the student.

PICTURE THE PROBLEM

The initial and final configurations of the student are shown in our sketch. Clearly, the mass distribution in the final configuration, with the masses held closer to the axis of rotation, results in a smaller moment of inertia.



STRATEGY

Ignoring friction in the axis of the stool, since none was mentioned, we conclude that no external torques act on the system. As a result, the angular momentum is conserved. Therefore, setting the initial angular momentum, $L_i = I_i\omega_i$, equal to the final angular momentum, $L_f = I_f\omega_f$, yields the final angular speed.

SOLUTION

Part (a)

1. Apply angular momentum conservation to this system: $L_i = L_f$

   $I_i\omega_i = I_f\omega_f$

2. Solve for the final angular speed, ω_f: $\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i$

3. Substitute numerical values: $\omega_f = \left(\frac{5.33 \text{ kg} \cdot \text{m}^2}{1.60 \text{ kg} \cdot \text{m}^2}\right)(3.72 \text{ rad/s}) = 12.4 \text{ rad/s}$

Part (b)

4. Use $L = I\omega$ to calculate the angular momentum. Substitute both initial and final values as a check:

   $L_i = I_i\omega_i = (5.33 \text{ kg} \cdot \text{m}^2)(3.72 \text{ rad/s}) = 19.8 \text{ kg} \cdot \text{m}^2/\text{s}$

   $L_f = I_f\omega_f = (1.60 \text{ kg} \cdot \text{m}^2)(12.4 \text{ rad/s}) = 19.8 \text{ kg} \cdot \text{m}^2/\text{s}$

INSIGHT

Initially the student completes one revolution roughly every two seconds. After pulling the weights in, the student's rotation rate has increased to almost two revolutions a second—quite a dizzying pace. The same physics applies to a rotating diver or a spinning ice skater.

PRACTICE PROBLEM

What moment of inertia would be required to give a final spin rate of 10.0 rad/s? [**Answer:** $I_f = (\omega_i/\omega_f)I_i = 1.99 \text{ kg} \cdot \text{m}^2$]

Some related homework problems: Problem 65, Problem 67, Problem 74

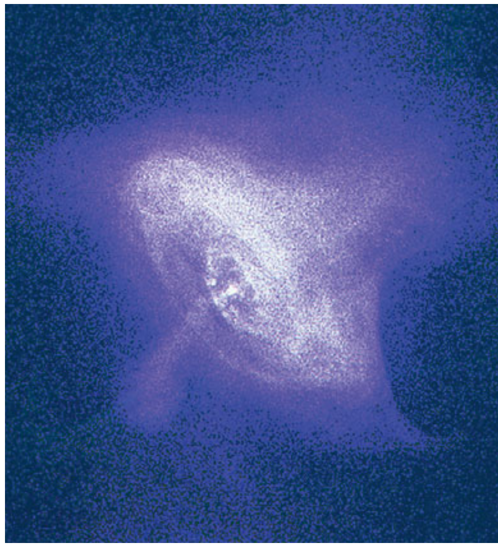
▲ This 1992 satellite photo of Hurricane Andrew (left), one of the most powerful hurricanes of recent decades, clearly suggests the rotating structure of the storm. The violence of the hurricane winds can be attributed in large part to conservation of angular momentum: as air is pushed inward toward the low pressure near the eye of the storm, its rotational velocity increases. The same principle, operating on a smaller scale, explains the tremendous destructive power of tornadoes. The tornado shown at right passed through downtown Miami on May 12, 1997.

REAL-WORLD PHYSICS

Hurricanes and tornadoes

An increasing angular speed, as experienced by the student in Example 11–9, can be observed in nature as well. For example, a hurricane draws circulating air in at ground level toward its "eye," where it then rises to an altitude of 10 miles or more. As air moves inward toward the axis of rotation, its angular speed increases, just as the masses held by the student speed up when they are pulled inward. For example, if the wind has a speed of only 3.0 mph at a distance of 300 miles from the center of the hurricane, it would speed up to 150 mph when it comes to within 6.0 miles of the center. Of course, this analysis ignores friction, which would certainly decrease the wind speed. Still, the basic principle—that a decreasing distance from the axis of rotation implies an increasing speed—applies to both the student and the hurricane. Similar behavior is observed in tornadoes and waterspouts.

Another example of conservation of angular momentum occurs in stellar explosions. On occasion a star will explode, sending a portion of its material out into space. After the explosion, the star collapses to a fraction of its original size, speeding up its rotation in the process. If the mass of the star is greater than 1.44 times the mass of the Sun, the collapse can continue until a *neutron star* is formed, with a radius of only about 10 to 20 km. Neutron stars have incredibly high densities; in fact, if you could bring a teaspoonful of neutron star material to the Earth, it would weigh about 100 million tons! In addition, neutron stars produce powerful beams of X-rays and other radiation that sweep across the sky like a gigantic lighthouse beam as the star rotates. On the Earth we see pulses of radiation from these rotating beams, one for each revolution of the star. These "pulsating stars," or *pulsars*, typically have periods ranging from about 2 ms to nearly 1 s. The Crab nebula (see Problems 9 and 106 in Chapter 10) is a famous example of such a system. The dependence of angular speed on radius for a collapsing star is considered in Active Example 11–4.



▲ Among the fastest rotating objects known in nature are pulsars: stars that have collapsed to a tiny fraction of their original size. Since all the angular momentum of a star must be conserved when it collapses, the dramatic decrease in radius is accompanied by a correspondingly great increase in rotational speed. The Crab nebula pulsar, the remains of a star whose explosion was observed on Earth nearly 1000 years ago, spins at about 30 rev/s. This X-ray photograph shows rings and jets of high-energy particles flying outward from the whirling neutron star at the center.

ACTIVE EXAMPLE 11–4 A STELLAR PERFORMANCE: FIND THE ANGULAR SPEED

A star of radius $R = 2.3 \times 10^8$ m rotates with an angular speed $\omega = 2.4 \times 10^{-6}$ rad/s. If this star collapses to a radius of 20.0 km, find its final angular speed. (Treat the star as if it were a uniform sphere, and assume that no mass is lost as the star collapses.)

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Apply conservation of angular momentum: $I_i\omega_i = I_f\omega_f$

CONTINUED ON NEXT PAGE

REAL-WORLD PHYSICS
Angular speed of a pulsar

CONTINUED FROM PREVIOUS PAGE

2. Write expressions for the initial and final moments of inertia: $I_i = \frac{2}{5}MR_i^2$ and $I_f = \frac{2}{5}MR_f^2$
3. Solve for the final angular speed: $\omega_f = (I_i/I_f)\omega_i = (R_i^2/R_f^2)\omega_i$
4. Substitute numerical values: $\omega_f = 320 \text{ rad/s}$

INSIGHT

The final angular speed corresponds to a period of about 20 ms, a typical period for pulsars. Since 320 rad/s is roughly 3000 rpm, it follows that a pulsar, which has the mass of a star, rotates as fast as the engine in a racing car.

YOUR TURN

At what radius will the star's period of rotation be equal to 15 ms?

(Answers to **Your Turn** *problems are given in the back of the book.)*

Note that if the student in Example 11–9 were to stretch his arms back out again, the resulting *increase* in the moment of inertia would cause a *decrease* in his angular speed. The same effect might apply to the Earth one day. For example, a melting of the polar ice caps would lead to an increase in the Earth's moment of inertia (as we saw in Chapter 10) and thus, by angular momentum conservation, the angular speed of the Earth would decrease. This would mean that more time would be required for the Earth to complete a revolution about its axis of rotation; that is, the day would lengthen.

Since angular momentum is conserved in the systems we have studied so far, it is natural to ask whether the energy is conserved as well. We consider this question in the next Conceptual Checkpoint.

CONCEPTUAL CHECKPOINT 11–4

COMPARE KINETIC ENERGIES

A skater pulls in her arms, decreasing her moment of inertia by a factor of two, and doubling her angular speed. Is her final kinetic energy **(a)** equal to, **(b)** greater than, or **(c)** less than her initial kinetic energy?

REASONING AND DISCUSSION

Let's calculate the initial and final kinetic energies, and compare them. The initial kinetic energy is

$$K_i = \tfrac{1}{2}I_i\omega_i^2$$

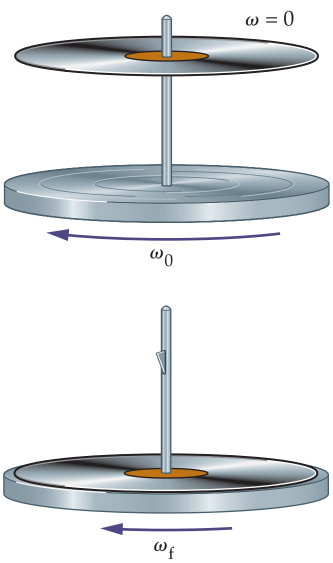
After pulling in her arms, the skater has half the moment of inertia and twice the angular speed. Hence, her final kinetic energy is

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}(I_i/2)(2\omega_i)^2 = 2\left(\tfrac{1}{2}I_i\omega_i^2\right) = 2K_i$$

Thus, the fact that K depends on the square of ω leads to an increase in the kinetic energy. The source of this additional energy is the work done by the muscles in the skater's arms as she pulls them in to her body.

ANSWER

(b) The skater's kinetic energy increases.



▲ **FIGURE 11–14 A rotational collision**

A nonrotating record dropped onto a rotating turntable is an example of a "rotational collision." Since only internal forces are involved during the collision, the final angular momentum is equal to the initial angular momentum.

Rotational Collisions

In the not-too-distant past, a person would play music by placing a record on a rotating turntable. Suppose, for example, that a turntable with a moment of inertia I_t is rotating freely with an initial angular speed ω_0. A record, with a moment of inertia I_r and initially at rest, is dropped straight down onto the rotating turntable, as in **Figure 11–14**. When the record lands, frictional forces between it and the turntable cause the record to speed up and the turntable to slow down, until they both have the same angular speed. Since only internal forces are involved during this process, it follows that the system's angular momentum is conserved. We can think of this event, then, as a "rotational collision."

Before the collision, the angular momentum of the system is

$$L_i = I_t\omega_0$$

After the collision, when both the record and the turntable are rotating with the angular speed ω_f, the system's angular momentum is

$$L_f = I_t\omega_f + I_r\omega_f$$

Setting $L_f = L_i$ yields the final angular speed:

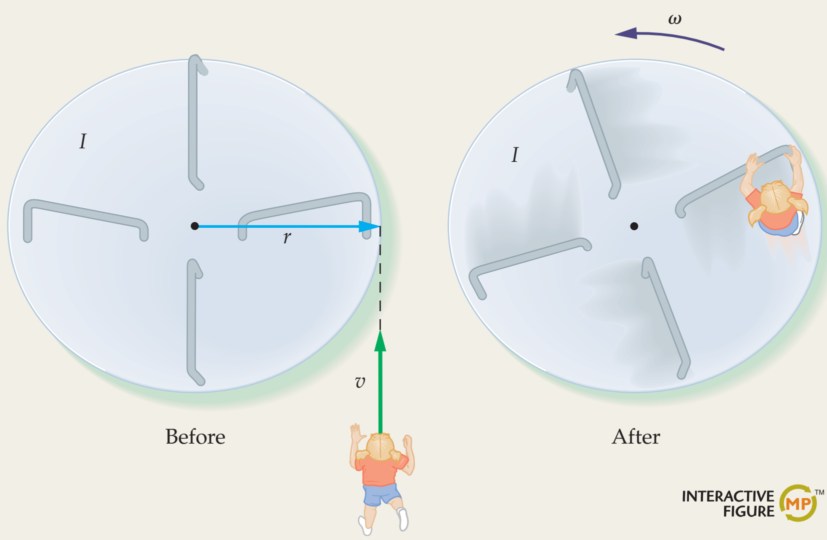
$$\omega_f = \left(\frac{I_t}{I_t + I_r}\right)\omega_0 \qquad 11\text{–}16$$

Since this collision is completely inelastic, we expect the final kinetic energy to be less than the initial kinetic energy.

We conclude this section with a somewhat different example of a rotational collision. The physical principles involved are precisely the same, however.

ACTIVE EXAMPLE 11–5 CONSERVE ANGULAR MOMENTUM: FIND THE ANGULAR SPEED

A 34.0-kg child runs with a speed of 2.80 m/s tangential to the rim of a stationary merry-go-round. The merry-go-round has a moment of inertia of 512 kg · m^2 and a radius of 2.31 m. When the child jumps onto the merry-go-round, the entire system begins to rotate. What is the angular speed of the system?



SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Write the initial angular momentum of the child: $L_i = rmv$
2. Write the final angular momentum of the system: $L_f = (I + mr^2)\omega$
3. Set $L_f = L_i$ and solve for the angular speed: $\omega = rmv/(I + mr^2)$
4. Substitute numerical values: $\omega = 0.317$ rad/s

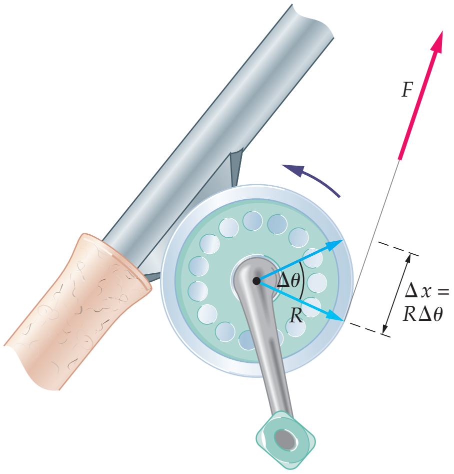
INSIGHT

If the moment of inertia of the merry-go-round had been zero, $I = 0$, the angular speed would be $\omega = v/r$. This means that the linear speed of the child, $r\omega = v$, is unchanged. If $I > 0$, however, the linear speed of the child is decreased. In this particular case, the child's linear speed after the collision is only $v = r\omega = 0.733$ m/s.

YOUR TURN

What initial speed does the child have if, after landing on the merry-go-round, it takes her 22.5 s to complete one revolution?

(Answers to **Your Turn** *problems are given in the back of the book.)*



▲ **FIGURE 11–15 Rotational work**
A force F pulling a length of line Δx from a fishing reel does the work $W = F\,\Delta x$. In terms of torque and angular displacement, the work can be expressed as $W = \tau\,\Delta\theta$.

The initial and final kinetic energies of the system in Active Example 11–5 are considered in Problem 66.

11–8 Rotational Work and Power

Just as a force acting through a distance performs work on an object, so too does a torque acting through an angular displacement. To see this, consider again the fishing line pulled from a reel. If the line is pulled with a force F for a distance Δx, as in **Figure 11–15**, the work done on the reel is

$$W = F\Delta x$$

Now, since the line is unwinding without slipping, it follows that the linear displacement of the line, Δx, is related to the angular displacement of the reel, $\Delta\theta$, by the following relation:

$$\Delta x = R\Delta\theta$$

In this equation, R is the radius of the reel, and $\Delta\theta$ is measured in radians. Thus, the work can be written as

$$W = F\Delta x = FR\Delta\theta$$

Finally, the torque exerted on the reel by the line is $\tau = RF$, and hence the work done on the reel is simply torque times angular displacement:

Work Done by Torque

$$W = \tau\Delta\theta \qquad \text{11–17}$$

Note again the analogies between angular and linear quantities in $W = F\Delta x$ and $W = \tau\Delta\theta$. As usual, τ is the analogue of F, and θ is the analogue of x.

As we saw in Chapter 7, the net work done on an object is equal to the change in its kinetic energy. This is the work–energy theorem:

$$W = \Delta K = K_f - K_i \qquad \text{11–18}$$

The work-energy theorem applies regardless of whether the work is done by a force acting through a distance or a torque acting through an angle.

Similarly, power is the amount of work done in a given time, regardless whether the work is done by a force or a torque. In the case of a torque, we have $W = \tau\Delta\theta$, and hence

Power Produced by a Torque

$$P = \frac{W}{\Delta t} = \tau\frac{\Delta\theta}{\Delta t} = \tau\omega \qquad \text{11–19}$$

Again, the analogy is clear between $P = Fv$ for the linear case, and $P = \tau\omega$ for the rotational case.

EXERCISE 11–5

It takes a good deal of effort to make homemade ice cream. **(a)** If the torque required to turn the handle on an ice cream maker is 5.7 N·m, how much work is expended on each complete revolution of the handle? **(b)** How much power is required to turn the handle if each revolution is completed in 1.5 s?

SOLUTION

a. Applying Equation 11–17 yields

$$W = \tau\Delta\theta = (5.7\ \text{N}\cdot\text{m})(2\pi\ \text{rad}) = 36\ \text{J}$$

b. Power is the work per time; that is,

$$P = W/\Delta t = (36\ \text{J})/(1.5\ \text{s}) = 24\ \text{W}$$

Equivalently, the angular speed of the handle is $\omega = (2\pi)/T = (2\pi)/(1.5\ \text{s}) = 4.2\ \text{rad/s}$, and therefore Equation 11–19 yields $P = \tau\omega = (5.7\ \text{N}\cdot\text{m})(4.2\ \text{rad/s}) = 24\ \text{W}$.

*11–9 The Vector Nature of Rotational Motion

We have mentioned many times that the angular velocity is a vector, and that we must be careful to use the proper sign for ω. But if the angular velocity is a vector, what is its direction?

To address this question, consider the rotating wheel shown in **Figure 11–16**. Each point on the rim of this wheel has a velocity vector pointing in a different direction in the plane of rotation. Since different parts of the wheel move in different directions, how can we assign a single direction to the angular velocity vector, $\vec{\boldsymbol{\omega}}$? The answer is that there is only one direction that remains fixed as the wheel rotates; the direction of the axis of rotation. By definition, then, the angular velocity vector, $\vec{\boldsymbol{\omega}}$, is taken to point along the axis of rotation.

Given that $\vec{\boldsymbol{\omega}}$ points along the axis of rotation, we must still decide whether it points to the left or to the right in Figure 11–16. The convention we use for assigning the direction of $\vec{\boldsymbol{\omega}}$ is referred to as the right-hand rule:

> **Right-Hand Rule for the Angular Velocity, $\vec{\boldsymbol{\omega}}$**
> Curl the fingers of the right hand in the direction of rotation.
> The thumb now points in the direction of the angular velocity, $\vec{\boldsymbol{\omega}}$.

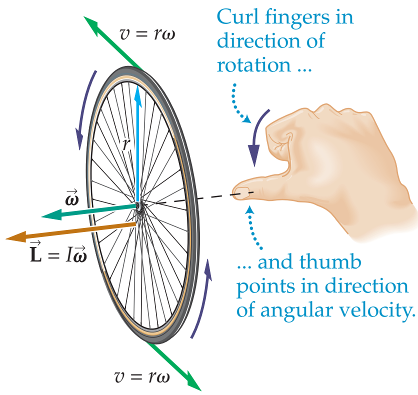


FIGURE 11–16 The right-hand rule for angular velocity
The angular velocity, $\vec{\boldsymbol{\omega}}$, of a rotating wheel points along the axis of rotation. Its direction is given by the right-hand rule.

The right-hand rule for $\vec{\boldsymbol{\omega}}$ is illustrated in Figure 11–16.

The same convention for direction applies to the angular momentum vector. First, recall that the angular momentum has a magnitude given by $L = I\omega$. Hence, we choose the direction of $\vec{\mathbf{L}}$ to be the same as the direction of $\vec{\boldsymbol{\omega}}$. That is

$$\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}} \qquad 11\text{–}20$$

The angular momentum vector is also illustrated in Figure 11–16.

Similarly, torque is a vector, and it too is defined to point along the axis of rotation. The right-hand rule for torque is similar to that for angular velocity:

> **Right-Hand Rule for Torque, $\vec{\boldsymbol{\tau}}$**
> Curl the fingers of the right hand in the direction of rotation that this torque would cause if it acted alone.
> The thumb now points in the direction of the torque vector, $\vec{\boldsymbol{\tau}}$.

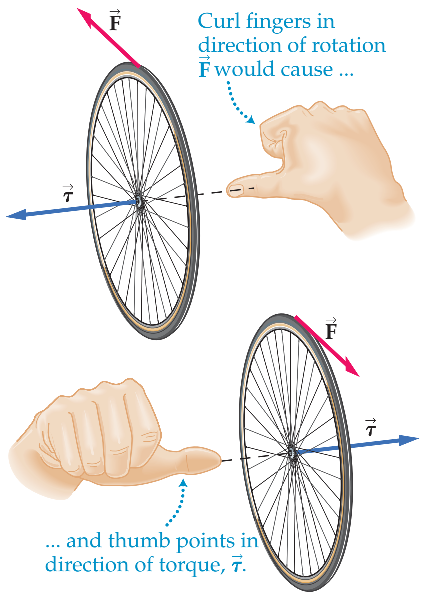


FIGURE 11–17 The right-hand rule for torque
Examples of torque vectors obtained using the right-hand rule.

Examples of torque vectors are given in **Figure 11–17**.

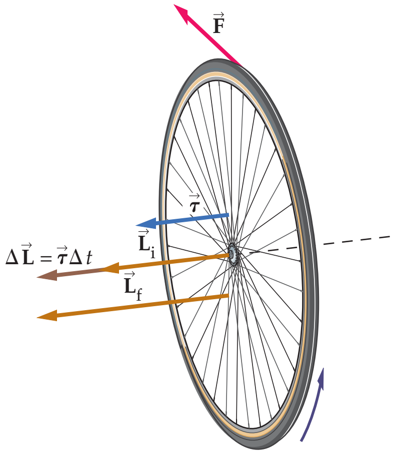
As an example of torque and angular momentum vectors, consider the spinning bicycle wheel shown in **Figure 11–18**. The angular momentum vector for the wheel points to the left, along the axis of rotation. If a person pushes on the rim of the wheel in the direction indicated, the resulting torque is also to the left, as shown in the figure. If this torque lasts for a time Δt, the angular momentum changes by the amount

$$\Delta\vec{\mathbf{L}} = \vec{\boldsymbol{\tau}}\,\Delta t$$

Adding $\Delta\vec{\mathbf{L}}$ to the original angular momentum $\vec{\mathbf{L}}_i$ yields the final angular momentum, $\vec{\mathbf{L}}_f$, shown in Figure 11–18. Since $\vec{\mathbf{L}}_f$ is in the same direction as $\vec{\mathbf{L}}_i$, but with a greater magnitude, it follows that the wheel is spinning in the same direction as

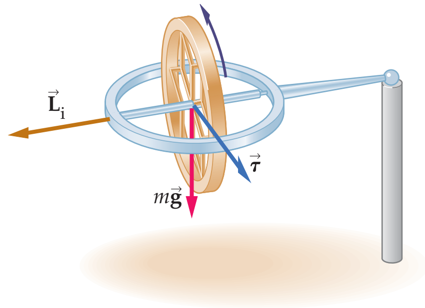
Children have always been fascinated by tops—but not only children. The physicists in the photo at right, Wolfgang Pauli and Niels Bohr, seem as delighted by a spinning top as any child. Their contributions to modern physics, discussed in Chapter 30, helped to show that subatomic particles, the ultimate constituents of matter, have a property (now referred to as "spin") that is in some ways analogous to the rotation of a top or a gyroscope.

▲ The 1.5-inch fused quartz sphere shown here is no ordinary ball. In fact, it is the most perfect sphere ever manufactured. If the Earth were this smooth, the change in elevation from the deepest ocean trench to the highest mountain peak would be only 16 feet. Such precision is required because this sphere is designed to serve as the rotor for an extremely sensitive gyroscope. The device, a million times more sensitive than those used in the best inertial navigation systems, orbits the Earth as part of an experiment to test predictions of Einstein's theory of general relativity.



▲ **FIGURE 11–18 Torque and angular momentum vectors**

A tangential push on the spinning wheel in the direction shown causes a torque to the left. As a result, the angular momentum increases. Hence, the wheel spins faster, as expected.



▲ **FIGURE 11–19 The torque exerted on a gyroscope**

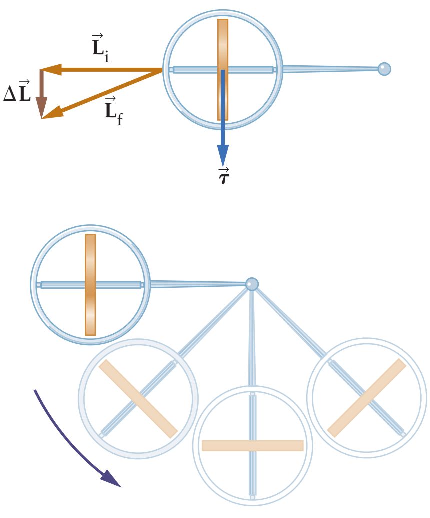
A spinning gyroscope has an initial angular momentum to the left. The torque due to gravity is out of the page.

before, only faster. This is to be expected, considering the direction of the person's push on the wheel.

On the other hand, if a person pushes on the wheel in the opposite direction, the torque vector points to the right. As a result, $\Delta\vec{L}$ points to the right as well. When we add $\Delta\vec{L}$ and $\vec{L}_i$ to obtain the final angular momentum, $\vec{L}_f$, we find that it has the same direction as $\vec{L}_i$, but a smaller magnitude. Hence, we conclude that the wheel spins more slowly, as one would expect.

Finally, a case of considerable interest is when the torque and angular momentum vectors are at right angles to one another. The classic example of such a system is the **gyroscope**. To begin, consider a gyroscope whose axis of rotation is horizontal, as in **Figure 11–19**. If the gyroscope were to be released with no spin it would simply fall, rotating counterclockwise downward about its point of support. Curling the fingers of the right hand in the counterclockwise direction, we see that the thumb, and hence the torque due to gravity, points out of the page.

Next, imagine the gyroscope to be spinning rapidly—as would normally be the case—with its angular momentum pointing to the left in Figure 11–19. If the gyroscope is released now, it doesn't fall as before, even though the torque is the same. To see what happens instead, consider the change in angular momentum, $\Delta\vec{L}$, caused by the torque, $\vec{\tau}$, acting for a small interval of time. As shown in **Figure 11–20**, the small change, $\Delta\vec{L}$, is at right angles to $\vec{L}_i$; hence the final angular momentum, $\vec{L}_f$, is essentially the same length as $\vec{L}_i$, but pointing in a direction slightly out of the page. With each small interval of time, the angular momentum vector continues to change in direction so that, viewed from above as in Figure 11–20, the gyroscope as a whole rotates in a counterclockwise sense around its support point. This type of motion, where the axis of rotation changes direction with time, is referred to as **precession**.



▲ **FIGURE 11–20 Precession of a gyroscope**

The gyroscope as viewed from above. In a time Δt the angular momentum changes by the amount $\Delta\vec{L} = \vec{\tau}\Delta t$. This causes the angular momentum vector, and hence the gyroscope as a whole, to rotate in a counterclockwise direction.

REAL-WORLD PHYSICS
Precession of the Earth

Because of its spinning motion about its rotational axis, the Earth may be considered as one rather large gyroscope. Gravitational forces exerted on the Earth by the Sun and the Moon subject it to external torques that cause its rotational axis to precess. At the moment, the rotational axis of the Earth points toward Polaris, the "North Star," which remains almost fixed in position in time-lapse photographs while the other stars move in circular paths about it. In a few hundred years, however, Polaris will also move in a circular path in the sky because the Earth's axis of rotation will point in a different direction. After 26,000 years the Earth will complete one full cycle of precession, and Polaris will again be the pole star.

REAL-WORLD PHYSICS
Gyroscopes in navigation and space

On a smaller scale, gyroscopes are used in the navigational systems of a variety of vehicles. In such applications, the rapidly spinning wheel of a gyroscope is mounted on nearly frictionless bearings so that it is practically free from external torques. If no torque acts on the gyroscope, its angular momentum vector remains unchanged both in magnitude and—here is the important point—in direction. With the axis of its gyroscope always pointing in the same, known direction, it is possible for a vehicle to maintain a desired direction of motion relative to the gyroscope's reference direction. On the Hubble Space Telescope, for example, six gyroscopes are used for pointing and stability, though it can operate with only three working gyroscopes if necessary.

THE BIG PICTURE PUTTING PHYSICS IN CONTEXT

LOOKING BACK

The concept of force (Chapters 5 and 6) is extended to torque, its rotational equivalent, in Section 11–1. We also apply Newton's laws to rotation in Section 11–6, just as for linear motion in Chapters 5 and 6.

The connection between rotational and linear quantities (Chapter 10) is used in Section 11–2 to relate torque to angular acceleration. In addition, we extend linear momentum (Chapter 9) to angular momentum in Sections 11–6 and 11–7.

Work and kinetic energy (Chapter 7) are applied to rotational systems in Section 11–8.

LOOKING AHEAD

Angular momentum and the conservation of angular momentum play important roles in the study of gravity. See, in particular, the discussion of Kepler's third law in Section 12–3.

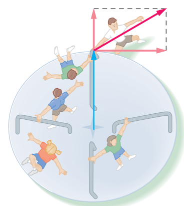
Torque arises in the discussion of magnetic fields and the forces they exert. See Section 22–5 in particular. The torques due to magnetic fields are also the key element in the operation of electric motors, as we see in Section 23–6.

Angular momentum is quantized (given discrete values) in the Bohr model of the hydrogen atom in Section 31–4.

CHAPTER SUMMARY

11–1 TORQUE

A force applied so as to cause an angular acceleration is said to exert a torque, τ.



Tangential Force
A force is tangential if it is tangent to a circle centered on the axis of rotation.

Torque Due to a Tangential Force
A tangential force F applied at a distance r from the axis of rotation produces a torque

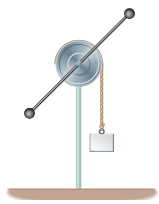
$$\tau = rF \qquad \text{11–1}$$

Torque for a General Force
A force exerted at an angle θ with respect to the radial direction, and applied at a distance r from the axis of rotation, produces the torque

$$\tau = rF\sin\theta \qquad \text{11–2}$$

11–2 TORQUE AND ANGULAR ACCELERATION

A single torque applied to an object gives it an angular acceleration.



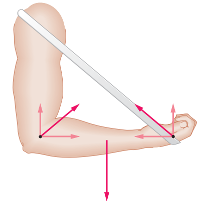
Newton's Second Law for Rotation
The connection between torque and angular acceleration is

$$\sum \tau = I\alpha \qquad \text{11–4}$$

In this expression, I is the moment of inertia about the axis of rotation and α is the angular acceleration about this axis.

Rotational/Translational Analogies
Torque is analogous to force, the moment of inertia is analogous to mass, and the angular acceleration is analogous to linear acceleration. Therefore, the rotational analogue of $F = ma$ is $\tau = I\alpha$.

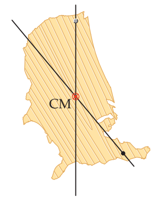
11–3 ZERO TORQUE AND STATIC EQUILIBRIUM



The conditions for an object to be in static equilibrium are that the total force and the total torque acting on the object must be zero:

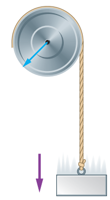
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum \tau = 0$$

11–4 CENTER OF MASS AND BALANCE



An object balances when it is supported at its center of mass.

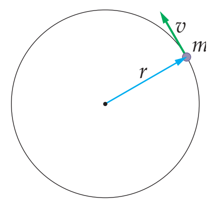
11–5 DYNAMIC APPLICATIONS OF TORQUE



Newton's second law can be applied to rotational systems in a way that is completely analogous to its application to linear systems.

Systems Involving Both Rotational and Linear Elements
In a system with both rotational and linear motions—such as a string passing over a pulley and attached to a mass—Newton's second law must be applied separately to the rotational and linear motions of the system. Connections between the two motions, such as $\alpha = a/r$, can be used to solve for all the accelerations in the system.

11–6 ANGULAR MOMENTUM



A moving object has angular momentum as long as its direction of motion does not extend through the axis of rotation.

Angular Momentum and Angular Speed
Angular momentum can be expressed in terms of angular speed and the moment of inertia as follows:

$$L = I\omega \qquad 11\text{–}11$$

This is the rotational analogue of $p = mv$.

Tangential Motion
An object of mass m moving tangentially with a speed v at a distance r from the axis of rotation has an angular momentum, L, given by

$$L = rmv \qquad 11\text{–}12$$

General Motion
If an object of mass m is a distance r from the axis of rotation and moves with a speed v at an angle θ with respect to the radial direction, its angular momentum is

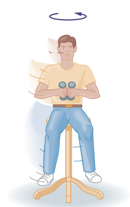
$$L = rmv \sin \theta \qquad 11\text{–}13$$

Newton's Second Law
Newton's second law can be expressed in terms of the rate of change of the angular momentum:

$$\sum \tau = I\alpha = \frac{\Delta L}{\Delta t} \qquad 11\text{–}14$$

This is the rotational analogue of $\Sigma F = \Delta p/\Delta t$.

11–7 CONSERVATION OF ANGULAR MOMENTUM



If the net external torque acting on a system is zero, its angular momentum is conserved:

$$L_f = L_i$$

Rotational Collisions
Systems in which two rotational objects come into contact can be thought of in terms of a "rotational collision." In such a case, the total angular momentum of the system is conserved.

11–8 ROTATIONAL WORK AND POWER

A torque acting through an angle does work, just as does a force acting through a distance.

Work Done by a Torque
A torque τ acting through an angle $\Delta\theta$ does a work W given by

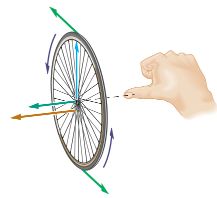
$$W = \tau\Delta\theta \qquad 11\text{–}17$$

Work–Energy Theorem
The work–energy theorem is

$$W = \Delta K = K_f - K_i \qquad 11\text{–}18$$

This theorem applies whether the work is done by a force or by a torque. In the linear case the kinetic energy is $\frac{1}{2}mv^2$; in the rotational case, the kinetic energy is $K = \frac{1}{2}I\omega^2$ (Equation 10–17).

*11–9 THE VECTOR NATURE OF ROTATIONAL MOTION



Rotational quantities have directions that point along the axis of rotation. The precise direction is given by the right-hand rule.

Right-Hand Rule
If the fingers of the *right hand* are curled in the direction of rotation, the thumb points in the direction of the rotational quantity in question. This rule applies to the angular velocity vector, $\vec{\boldsymbol{\omega}}$, the angular acceleration vector, $\vec{\boldsymbol{\alpha}}$, the angular momentum vector, $\vec{\mathbf{L}}$, and the torque vector, $\vec{\boldsymbol{\tau}}$.

PROBLEM-SOLVING SUMMARY

Type of Problem	Relevant Physical Concepts	Related Examples
Find the torque exerted on a system.	The torque exerted by a tangential force a distance r from the axis of rotation is $\tau = rF$. If the force is at an angle θ to the radial direction, the torque is $\tau = rF\sin\theta$.	Example 11–1
Determine the angular acceleration of a system.	First, calculate the torque exerted on the system. Next, find the angular acceleration using Newton's second law as applied to rotation, namely, $\tau = I\alpha$.	Examples 11–2, 11–3
Find the forces required for static equilibrium.	Static equilibrium requires that both the net force and the net torque acting on a system be zero.	Examples 11–4, 11–5, 11–6 Active Examples 11–1, 11–2, 11–3
Find the final angular momentum of a system.	A torque changes the angular momentum L of a system with time as follows: $\tau = \Delta L/\Delta t$. If no net torque acts on a system, its angular momentum is conserved.	Examples 11–8, 11–9 Active Examples 11–4, 11–5

CONCEPTUAL QUESTIONS

(Answers to odd-numbered Conceptual Questions can be found in the back of the book.)

1. Two forces produce the same torque. Does it follow that they have the same magnitude? Explain.
2. A car pitches down in front when the brakes are applied sharply. Explain this observation in terms of torques.
3. A tightrope walker uses a long pole to aid in balancing. Why?
4. When a motorcycle accelerates rapidly from a stop it sometimes "pops a wheelie"; that is, its front wheel may lift off the ground. Explain this behavior in terms of torques.
5. Give an example of a system in which the net torque is zero but the net force is nonzero.
6. Give an example of a system in which the net force is zero but the net torque is nonzero.
7. Is the normal force exerted by the ground the same for all four tires on your car? Explain.
8. Give two everyday examples of objects that are not in static equilibrium.
9. Give two everyday examples of objects that are in static equilibrium.
10. Can an object have zero translational acceleration and, at the same time, have nonzero angular acceleration? If your answer is no, explain why not. If your answer is yes, give a specific example.
11. Stars form when a large rotating cloud of gas collapses. What happens to the angular speed of the gas cloud as it collapses?
12. What purpose does the tail rotor on a helicopter serve?

13. Is it possible to change the angular momentum of an object without changing its linear momentum? If your answer is no, explain why not. If your answer is yes, give a specific example.

14. Suppose a diver springs into the air with no initial angular velocity. Can the diver begin to rotate by folding into a tucked position? Explain.

PROBLEMS AND CONCEPTUAL EXERCISES

Note: Answers to odd-numbered Problems and Conceptual Exercises can be found in the back of the book. **IP** *denotes an integrated problem, with both conceptual and numerical parts;* **BIO** *identifies problems of biological or medical interest;* **CE** *indicates a conceptual exercise.* **Predict/Explain** *problems ask for two responses:* **(a)** *your prediction of a physical outcome, and* **(b)** *the best explanation among three provided. On all problems, red bullets (•, ••, •••) are used to indicate the level of difficulty.*

SECTION 11–1 TORQUE

1. • To tighten a spark plug, it is recommended that a torque of 15 N · m be applied. If a mechanic tightens the spark plug with a wrench that is 25 cm long, what is the minimum force necessary to create the desired torque?

2. • **Pulling a Weed** The gardening tool shown in **Figure 11–21** is used to pull weeds. If a 1.23-N · m torque is required to pull a given weed, what force did the weed exert on the tool?

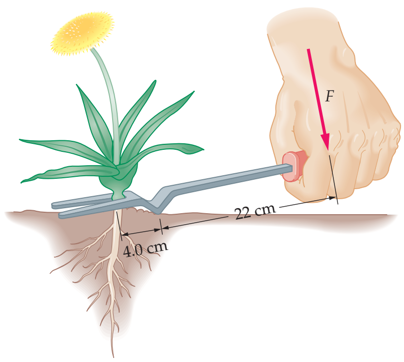


FIGURE 11–21 Problem 2

3. • A 1.61-kg bowling trophy is held at arm's length, a distance of 0.605 m from the shoulder joint. What torque does the trophy exert about the shoulder if the arm is **(a)** horizontal, or **(b)** at an angle of 22.5° below the horizontal?

4. • A person slowly lowers a 3.6-kg crab trap over the side of a dock, as shown in **Figure 11–22**. What torque does the trap exert about the person's shoulder?

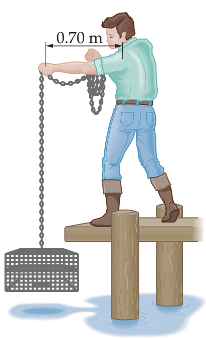


FIGURE 11–22 Problem 4

5. •• **IP BIO Force to Hold a Baseball** A person holds a 1.42-N baseball in his hand, a distance of 34.0 cm from the elbow joint, as shown in **Figure 11–23**. The biceps, attached at a distance of 2.75 cm from the elbow, exerts an upward force of 12.6 N on the forearm. Consider the forearm and hand to be a uniform rod with a mass of 1.20 kg. **(a)** Calculate the net torque acting on the forearm and hand. Use the elbow joint as the axis of rotation. **(b)** If the net torque obtained in part (a) is nonzero, in which direction will the forearm and hand rotate? **(c)** Would the torque exerted on the forearm by the biceps increase or decrease if the biceps were attached farther from the elbow joint?

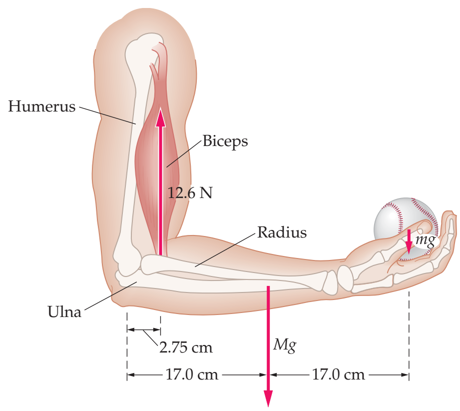


FIGURE 11–23 Problems 5 and 19

6. •• At the local playground, a 16-kg child sits on the end of a horizontal teeter-totter, 1.5 m from the pivot point. On the other side of the pivot an adult pushes straight down on the teeter-totter with a force of 95 N. In which direction does the teeter-totter rotate if the adult applies the force at a distance of **(a)** 3.0 m, **(b)** 2.5 m, or **(c)** 2.0 m from the pivot?

SECTION 11–2 TORQUE AND ANGULAR ACCELERATION

7. • **CE Predict/Explain** Consider the pulley–block systems shown in Conceptual Checkpoint 11–1. **(a)** Is the tension in the string on the left-hand rotating system greater than, less than, or equal to the weight of the mass attached to that string? **(b)** Choose the *best explanation* from among the following:

I. The mass is in free fall once it is released.
II. The string rotates the pulley in addition to supporting the mass.
III. The mass accelerates downward.

8. • **CE Predict/Explain** Consider the pulley–block systems shown in Conceptual Checkpoint 11–1. **(a)** Is the tension in the string on the left-hand rotating system greater than, less than, or equal to the tension in the string on the right-hand rotating system? **(b)** Choose the *best explanation* from among the following:

I. The mass in the right-hand system has the greater downward acceleration.
II. The masses are equal.

III. The mass in the left-hand system has the greater downward acceleration.

9. • **CE** Suppose a torque rotates your body about one of three different axes of rotation: case A, an axis through your spine; case B, an axis through your hips; and case C, an axis through your ankles. Rank these three axes of rotation in increasing order of the angular acceleration produced by the torque. Indicate ties where appropriate.

10. • A torque of 0.97 N · m is applied to a bicycle wheel of radius 35 cm and mass 0.75 kg. Treating the wheel as a hoop, find its angular acceleration.

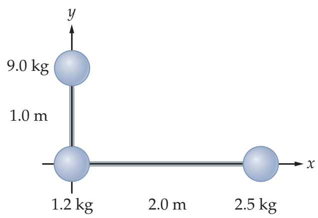
11. • When a ceiling fan rotating with an angular speed of 2.75 rad/s is turned off, a frictional torque of 0.120 N · m slows it to a stop in 22.5 s. What is the moment of inertia of the fan?

12. • When the play button is pressed, a CD accelerates uniformly from rest to 450 rev/min in 3.0 revolutions. If the CD has a radius of 6.0 cm and a mass of 17 g, what is the torque exerted on it?

13. •• A person holds a ladder horizontally at its center. Treating the ladder as a uniform rod of length 3.15 m and mass 8.42 kg, find the torque the person must exert on the ladder to give it an angular acceleration of 0.302 rad/s^2.

14. •• **IP** A wheel on a game show is given an initial angular speed of 1.22 rad/s. It comes to rest after rotating through 0.75 of a turn. **(a)** Find the average torque exerted on the wheel given that it is a disk of radius 0.71 m and mass 6.4 kg. **(b)** If the mass of the wheel is doubled and its radius is halved, will the angle through which it rotates before coming to rest increase, decrease, or stay the same? Explain. (Assume that the average torque exerted on the wheel is unchanged.)

15. •• The L-shaped object in **Figure 11–24** consists of three masses connected by light rods. What torque must be applied to this object to give it an angular acceleration of 1.20 rad/s^2 if it is rotated about **(a)** the x axis, **(b)** the y axis, or **(c)** the z axis (which is through the origin and perpendicular to the page)?

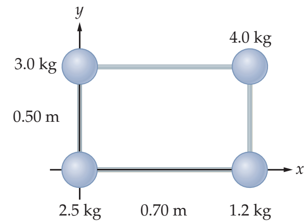


▲ **FIGURE 11–24** Problems 15, 16, and 82

16. •• **CE** The L-shaped object described in Problem 15 can be rotated in one of the following three ways: case A, about the x axis; case B, about the y axis; and case C, about the z axis (which passes through the origin perpendicular to the plane of the figure). If the same torque τ is applied in each of these cases, rank them in increasing order of the resulting angular acceleration. Indicate ties where appropriate.

17. •• **CE** A motorcycle accelerates from rest, and both the front and rear tires roll without slipping. **(a)** Is the force exerted by the ground on the rear tire in the forward or in the backward direction? Explain. **(b)** Is the force exerted by the ground on the front tire in the forward or in the backward direction? Explain. **(c)** If the moment of inertia of the front tire is increased, will the motorcycle's acceleration increase, decrease, or stay the same? Explain.

18. •• **IP** A torque of 13 N · m is applied to the rectangular object shown in **Figure 11–25**. The torque can act about the x axis, the y axis, or the z axis, which passes through the origin and points out of the page. **(a)** In which case does the object experience the greatest angular acceleration? The least angular acceleration? Explain. Find the angular acceleration when the torque acts about **(b)** the x axis, **(c)** the y axis, and **(d)** the z axis.



▲ **FIGURE 11–25** Problems 18 and 83

19. •• A fish takes the bait and pulls on the line with a force of 2.2 N. The fishing reel, which rotates without friction, is a cylinder of radius 0.055 m and mass 0.99 kg. **(a)** What is the angular acceleration of the fishing reel? **(b)** How much line does the fish pull from the reel in 0.25 s?

20. •• Repeat the previous problem, only now assume the reel has a friction clutch that exerts a restraining torque of 0.047 N · m.

SECTION 11–3 ZERO TORQUE AND STATIC EQUILIBRIUM

21. • **CE Predict/Explain** Suppose the person in Active Example 11–3 climbs higher on the ladder. **(a)** As a result, is the ladder more likely, less likely, or equally likely to slip? **(b)** Choose the *best explanation* from among the following:

I. The forces are the same regardless of the person's position.
II. The magnitude of f_2 must increase as the person moves upward.
III. When the person is higher, the ladder presses down harder on the floor.

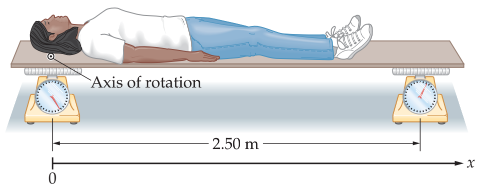
22. • A string that passes over a pulley has a 0.321-kg mass attached to one end and a 0.635-kg mass attached to the other end. The pulley, which is a disk of radius 9.40 cm, has friction in its axle. What is the magnitude of the frictional torque that must be exerted by the axle if the system is to be in static equilibrium?

23. • To loosen the lid on a jar of jam 8.9 cm in diameter, a torque of 8.5 N · m must be applied to the circumference of the lid. If a jar wrench whose handle extends 15 cm from the center of the jar is attached to the lid, what is the minimum force required to open the jar?

24. • Consider the system in Active Example 11–1, this time with the axis of rotation at the location of the child. Write out both the condition for zero net force and the condition for zero net torque. Solve for the two forces.

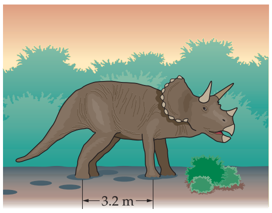
25. •• **IP BIO** Referring to the person holding a baseball in Problem 5, suppose the biceps exert just enough upward force to keep the system in static equilibrium. **(a)** Is the force exerted by the biceps more than, less than, or equal to the combined weight of the forearm, hand, and baseball? Explain. **(b)** Determine the force exerted by the biceps.

26. •• **IP BIO A Person's Center of Mass** To determine the location of her center of mass, a physics student lies on a lightweight plank supported by two scales 2.50 m apart, as indicated in **Figure 11–26**. If the left scale reads 290 N, and the right scale reads 122 N, find **(a)** the student's mass and **(b)** the distance from the student's head to her center of mass.



▲ **FIGURE 11–26** Problem 26

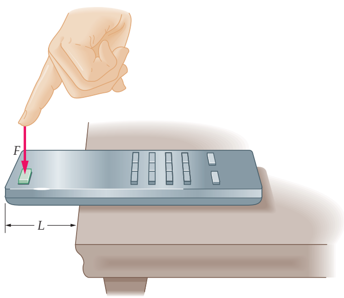
27. •• **Triceratops** A set of fossilized triceratops footprints discovered in Texas show that the front and rear feet were 3.2 m apart, as shown in **Figure 11–27**. The rear footprints were observed to be twice as deep as the front footprints. Assuming that the rear feet pressed down on the ground with twice the force exerted by the front feet, find the horizontal distance from the rear feet to the triceratops's center of mass.



▲ **FIGURE 11–27** Problem 27

28. •• **IP** A schoolyard teeter-totter with a total length of 5.2 m and a mass of 38 kg is pivoted at its center. A 19-kg child sits on one end of the teeter-totter. **(a)** Where should a parent push vertically downward with a force of 210 N in order to hold the teeter-totter level? **(b)** Where should the parent push with a force of 310 N? **(c)** How would your answers to parts (a) and (b) change if the mass of the teeter-totter were doubled? Explain.

29. •• A 0.122-kg remote control 23.0 cm long rests on a table, as shown in **Figure 11–28**, with a length L overhanging its edge. To operate the power button on this remote requires a force of 0.365 N. How far can the remote control extend beyond the edge of the table and still not tip over when you press the power button? Assume the mass of the remote is distributed uniformly, and that the power button is 1.41 cm from the overhanging end of the remote.



▲ **FIGURE 11–28** Problem 29

30. •• **IP** A 0.16-kg meterstick is held perpendicular to a vertical wall by a 2.5-m string going from the wall to the far end of the stick. **(a)** Find the tension in the string. **(b)** If a shorter string is used, will its tension be greater than, less than, or the same as that found in part (a)? **(c)** Find the tension in a 2.0-m string.

31. •• Repeat Example 11–4, this time with a uniform diving board that weighs 225 N.

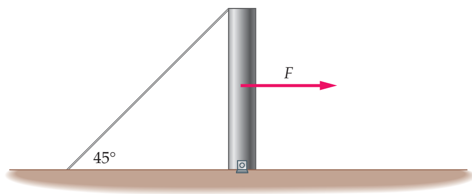
32. •• Babe Ruth steps to the plate and casually points to left center field to indicate the location of his next home run. The mighty Babe holds his bat across his shoulder, with one hand holding the small end of the bat. The bat is horizontal, and the distance from the small end of the bat to the shoulder is 22.5 cm. If the bat has a mass of 1.10 kg and has a center of mass that is 67.0 cm from the small end of the bat, find the magnitude and direction of the force exerted by **(a)** the hand and **(b)** the shoulder.

33. •• A uniform metal rod, with a mass of 3.7 kg and a length of 1.2 m, is attached to a wall by a hinge at its base. A horizontal wire bolted to the wall 0.51 m above the base of the rod holds the rod at an angle of 25° above the horizontal. The wire is attached to the top of the rod. **(a)** Find the tension in the wire. Find **(b)** the horizontal and **(c)** the vertical components of the force exerted on the rod by the hinge.

34. •• **IP** In the previous problem, suppose the wire is shortened, so that the rod now makes an angle of 35° with the horizontal. The wire is horizontal, as before. **(a)** Do you expect the tension in the wire to increase, decrease, or stay the same as a result of its new length? Explain. **(b)** Calculate the tension in the wire.

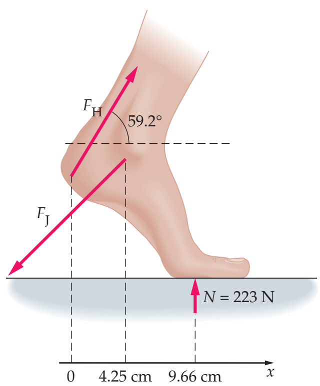
35. •• Repeat Active Example 11–3, this time with a uniform 7.2-kg ladder that is 4.0 m long.

36. •• A rigid, vertical rod of negligible mass is connected to the floor by a bolt through its lower end, as shown in **Figure 11–29**. The rod also has a wire connected between its top end and the floor. If a horizontal force F is applied at the midpoint of the rod, find **(a)** the tension in the wire, and **(b)** the horizontal and **(c)** the vertical components of force exerted by the bolt on the rod.



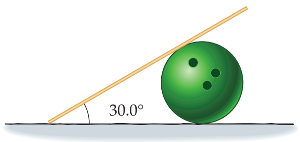
▲ **FIGURE 11–29** Problems 36, 111, and 112

37. ••• **BIO Forces in the Foot** **Figure 11–30** shows the forces acting on a sprinter's foot just before she takes off at the start of the race. Find the magnitude of the force exerted on the heel by the Achilles tendon, F_H, and the magnitude of the force exerted on the foot at the ankle joint, F_J.



▲ **FIGURE 11–30** Problem 37

38. ••• A stick with a mass of 0.214 kg and a length of 0.436 m rests in contact with a bowling ball and a rough floor, as shown in **Figure 11–31**. The bowling ball has a diameter of 21.6 cm, and the angle the stick makes with the horizontal is 30.0°. You may assume there is no friction between the stick and the bowling ball, though friction with the floor must be taken into account. **(a)** Find the magnitude of the force exerted on the stick by the bowling ball. **(b)** Find the horizontal component of the force exerted on the stick by the floor. **(c)** Repeat part (b) for the vertical component of the force.



▲ **FIGURE 11–31** Problem 38

39. ••• **IP** A uniform crate with a mass of 16.2 kg rests on a floor with a coefficient of static friction equal to 0.571. The crate is a uniform cube with sides 1.21 m in length. **(a)** What horizontal force applied to the top of the crate will initiate tipping? **(b)** If the horizontal force is applied halfway to the top of the crate, it will begin to slip before it tips. Explain.

40. ••• In the previous problem, **(a)** what is the minimum height where the force F can be applied so that the crate begins to tip before sliding? **(b)** What is the magnitude of the force in this case?

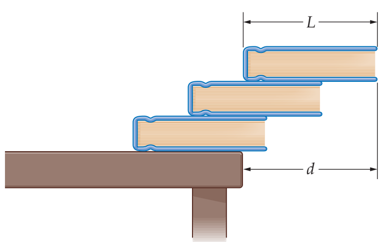
SECTION 11–4 CENTER OF MASS AND BALANCE

41. • A hand-held shopping basket 62.0 cm long has a 1.81-kg carton of milk at one end, and a 0.722-kg box of cereal at the other end. Where should a 1.80-kg container of orange juice be placed so that the basket balances at its center?

42. • If the cat in Active Example 11–2 has a mass of 2.8 kg, how close to the right end of the two-by-four can it walk before the board begins to tip?

43. •• **IP** A 0.34-kg meterstick balances at its center. If a necklace is suspended from one end of the stick, the balance point moves 9.5 cm toward that end. **(a)** Is the mass of the necklace more than, less than, or the same as that of the meterstick? Explain. **(b)** Find the mass of the necklace.

44. •• **Maximum Overhang** Three identical, uniform books of length L are stacked one on top the other. Find the maximum overhang distance d in **Figure 11–32** such that the books do not fall over.



▲ **FIGURE 11–32** Problems 44 and 107

45. •• A baseball bat balances 71.1 cm from one end. If a 0.560-kg glove is attached to that end, the balance point moves 24.7 cm toward the glove. Find the mass of the bat.

SECTION 11–5 DYNAMIC APPLICATIONS OF TORQUE

46. •• A 2.85-kg bucket is attached to a disk-shaped pulley of radius 0.121 m and mass 0.742 kg. If the bucket is allowed to fall, **(a)** what is its linear acceleration? **(b)** What is the angular acceleration of the pulley? **(c)** How far does the bucket drop in 1.50 s?

47. •• **IP** In the previous problem, **(a)** is the tension in the rope greater than, less than, or equal to the weight of the bucket? Explain. **(b)** Calculate the tension in the rope.

48. •• A child exerts a tangential 42.2-N force on the rim of a disk-shaped merry-go-round with a radius of 2.40 m. If the merry-go-round starts at rest and acquires an angular speed of 0.0860 rev/s in 3.50 s, what is its mass?

49. •• **IP** You pull downward with a force of 28 N on a rope that passes over a disk-shaped pulley of mass 1.2 kg and radius 0.075 m. The other end of the rope is attached to a 0.67-kg mass. **(a)** Is the tension in the rope the same on both sides of the pulley? If not, which side has the largest tension? **(b)** Find the tension in the rope on both sides of the pulley.

50. •• Referring to the previous problem, find the linear acceleration of the 0.67-kg mass.

51. ••• A uniform meterstick of mass M has an empty paint can of mass m hanging from one end. The meterstick and the can balance at a point 20.0 cm from the end of the stick where the can is attached. When the balanced stick–can system is suspended from a scale, the reading on the scale is 2.54 N. Find the mass of **(a)** the meterstick and **(b)** the paint can.

52. ••• **Atwood's Machine** An Atwood's machine consists of two masses, m_1 and m_2, connected by a string that passes over a pulley. If the pulley is a disk of radius R and mass M, find the acceleration of the masses.

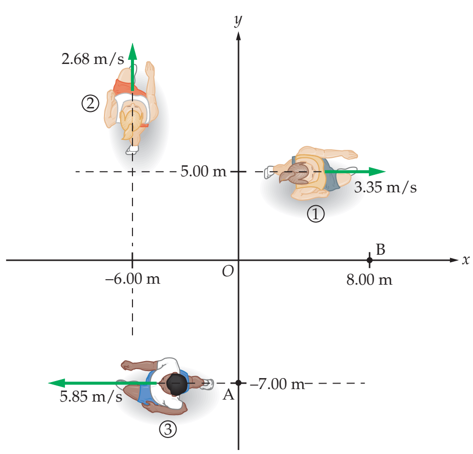
SECTION 11–6 ANGULAR MOMENTUM

53. • Calculate the angular momentum of the Earth about its own axis, due to its daily rotation. Assume that the Earth is a uniform sphere.

54. • A 0.015-kg record with a radius of 15 cm rotates with an angular speed of $33\frac{1}{3}$ rpm. Find the angular momentum of the record.

55. • In the previous problem, a 1.1-g fly lands on the rim of the record. What is the fly's angular momentum?

56. • Jogger 1 in **Figure 11–33** has a mass of 65.3 kg and runs in a straight line with a speed of 3.35 m/s. **(a)** What is the magnitude



▲ **FIGURE 11–33** Problems 56, 57, and 58 of the jogger's linear momentum? **(b)** What is the magnitude of the jogger's angular momentum with respect to the origin, *O*?

57. • Repeat the previous problem for the case of jogger 2, whose speed is 2.68 m/s and whose mass is 58.2 kg.

58. •• **IP** Suppose jogger 3 in Figure 11–33 has a mass of 62.2 kg and a speed of 5.85 m/s. **(a)** Is the magnitude of the jogger's angular momentum greater with respect to point A or point B? Explain. **(b)** Is the magnitude of the jogger's angular momentum with respect to point B greater than, less than, or the same as it is with respect to the origin, *O*? Explain. **(c)** Calculate the magnitude of the jogger's angular momentum with respect to points A, B, and *O*.

59. •• A torque of 0.12 N · m is applied to an egg beater. **(a)** If the egg beater starts at rest, what is its angular momentum after 0.65 s? **(b)** If the moment of inertia of the egg beater is 2.5×10^{-3} kg · m^2, what is its angular speed after 0.65 s?

60. •• A windmill has an initial angular momentum of 8500 kg · m^2/s. The wind picks up, and 5.86 s later the windmill's angular momentum is 9700 kg · m^2/s. What was the torque acting on the windmill, assuming it was constant during this time?

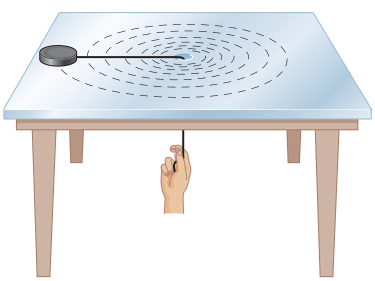
61. •• Two gerbils run in place with a linear speed of 0.55 m/s on an exercise wheel that is shaped like a hoop. Find the angular momentum of the system if each gerbil has a mass of 0.22 kg and the exercise wheel has a radius of 9.5 cm and a mass of 5.0 g.

SECTION 11–7 CONSERVATION OF ANGULAR MOMENTUM

62. • **CE Predict/Explain** A student rotates on a frictionless piano stool with his arms outstretched, a heavy weight in each hand. Suddenly he lets go of the weights, and they fall to the floor. As a result, does the student's angular speed increase, decrease, or stay the same? **(b)** Choose the *best explanation* from among the following:

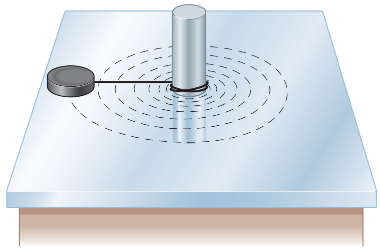
I. The loss of angular momentum when the weights are dropped causes the student to rotate more slowly.
II. The student's moment of inertia is decreased by dropping the weights.
III. Dropping the weights exerts no torque on the student, but the floor exerts a torque on the weights when they land.

63. • **CE** A puck on a horizontal, frictionless surface is attached to a string that passes through a hole in the surface, as shown in **Figure 11–34**. As the puck rotates about the hole, the string is pulled downward, bringing the puck closer to the hole. During this process, do the puck's **(a)** linear speed, **(b)** angular speed, and **(c)** angular momentum increase, decrease, or stay the same?



▲ **FIGURE 11–34** Problems 63 and 93

64. • **CE** A puck on a horizontal, frictionless surface is attached to a string that wraps around a pole of finite radius, as shown in **Figure 11–35**. **(a)** As the puck moves along the spiral path, does its speed increase, decrease, or stay the same? Explain. **(b)** Does its angular momentum increase, decrease, or stay the same? Explain.



▲ **FIGURE 11–35** Problem 64

65. • As an ice skater begins a spin, his angular speed is 3.17 rad/s. After pulling in his arms, his angular speed increases to 5.46 rad/s. Find the ratio of the skater's final moment of inertia to his initial moment of inertia.

66. • Calculate both the initial and the final kinetic energies of the system described in Active Example 11–5.

67. • A diver tucks her body in midflight, decreasing her moment of inertia by a factor of two. By what factor does her angular speed change?

68. •• **IP** In the previous problem, **(a)** does the diver's kinetic energy increase, decrease, or stay the same? **(b)** Calculate the ratio of the final kinetic energy to the initial kinetic energy for the diver.

69. •• A disk-shaped merry-go-round of radius 2.63 m and mass 155 kg rotates freely with an angular speed of 0.641 rev/s. A 59.4-kg person running tangential to the rim of the merry-go-round at 3.41 m/s jumps onto its rim and holds on. Before jumping on the merry-go-round, the person was moving in the same direction as the merry-go-round's rim. What is the final angular speed of the merry-go-round?

70. •• **IP** In the previous problem, **(a)** does the kinetic energy of the system increase, decrease, or stay the same when the person jumps on the merry-go-round? **(b)** Calculate the initial and final kinetic energies for this system.

71. •• A student sits at rest on a piano stool that can rotate without friction. The moment of inertia of the student–stool system is 4.1 kg · m^2. A second student tosses a 1.5-kg mass with a speed of 2.7 m/s to the student on the stool, who catches it at a distance of 0.40 m from the axis of rotation. What is the resulting angular speed of the student and the stool?

72. •• **IP** Referring to the previous problem, **(a)** does the kinetic energy of the mass–student–stool system increase, decrease, or stay the same as the mass is caught? **(b)** Calculate the initial and final kinetic energies of the system.

73. •• **IP** A turntable with a moment of inertia of 5.4×10^{-3} kg · m^2 rotates freely with an angular speed of $33\frac{1}{3}$ rpm. Riding on the rim of the turntable, 15 cm from the center, is a cute, 32-g mouse. **(a)** If the mouse walks to the center of the turntable, will the turntable rotate faster, slower, or at the same rate? Explain. **(b)** Calculate the angular speed of the turntable when the mouse reaches the center.

74. •• A student on a piano stool rotates freely with an angular speed of 2.95 rev/s. The student holds a 1.25-kg mass in each outstretched arm, 0.759 m from the axis of rotation. The combined moment of inertia of the student and the stool, ignoring the two masses, is 5.43 kg · m^2, a value that remains constant. **(a)** As the student pulls his arms inward, his angular speed increases to 3.54 rev/s. How far are the masses from the axis of rotation at this time, considering the masses to be points? **(b)** Calculate the initial and final kinetic energies of the system.

75. ••• **Walking on a Merry-Go-Round** A child of mass m stands at rest near the rim of a stationary merry-go-round of radius R and moment of inertia I. The child now begins to walk around the circumference of the merry-go-round with a tangential speed v with respect to the merry-go-round's surface. **(a)** What is the child's speed with respect to the ground? Check your result in the limits **(b)** $I \to 0$ and **(c)** $I \to \infty$.

SECTION 11–8 ROTATIONAL WORK AND POWER

76. • **CE Predict/Explain** Two spheres of equal mass and radius are rolling across the floor with the same speed. Sphere 1 is a uniform solid; sphere 2 is hollow. Is the work required to stop sphere 1 greater than, less than, or equal to the work required to stop sphere 2? **(b)** Choose the *best explanation* from among the following:
   I. Sphere 2 has the greater moment of inertia and hence the greater rotational kinetic energy.
   II. The spheres have equal mass and speed; therefore, they have the same kinetic energy.
   III. The hollow sphere has less kinetic energy.

77. • How much work must be done to accelerate a baton from rest to an angular speed of 7.4 rad/s about its center? Consider the baton to be a uniform rod of length 0.53 m and mass 0.44 kg.

78. • Turning a doorknob through 0.25 of a revolution requires 0.14 J of work. What is the torque required to turn the doorknob?

79. • A person exerts a tangential force of 36.1 N on the rim of a disk-shaped merry-go-round of radius 2.74 m and mass 167 kg. If the merry-go-round starts at rest, what is its angular speed after the person has rotated it through an angle of 32.5°?

80. • To prepare homemade ice cream, a crank must be turned with a torque of 3.95 N · m. How much work is required for each complete turn of the crank?

81. • **Power of a Dental Drill** A popular make of dental drill can operate at a speed of 42,500 rpm while producing a torque of 3.68 oz · in. What is the power output of this drill? Give your answer in watts.

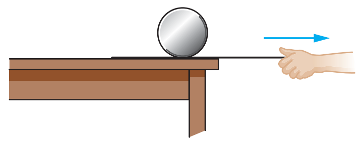
82. •• The L-shaped object in Figure 11–24 consists of three masses connected by light rods. Find the work that must be done on this object to accelerate it from rest to an angular speed of 2.35 rad/s about **(a)** the x axis, **(b)** the y axis, and **(c)** the z axis (which is through the origin and perpendicular to the page).

83. •• The rectangular object in Figure 11–25 consists of four masses connected by light rods. What power must be applied to this object to accelerate it from rest to an angular speed of 2.5 rad/s in 6.4 s about **(a)** the x axis, **(b)** the y axis, and **(c)** the z axis (which is through the origin and perpendicular to the page)?

84. •• **IP** A circular saw blade accelerates from rest to an angular speed of 3620 rpm in 6.30 revolutions. **(a)** Find the torque exerted on the saw blade, assuming it is a disk of radius 15.2 cm and mass 0.755 kg. **(b)** Is the angular speed of the saw blade after 3.15 revolutions greater than, less than, or equal to 1810 rpm? Explain. **(c)** Find the angular speed of the blade after 3.15 revolutions.

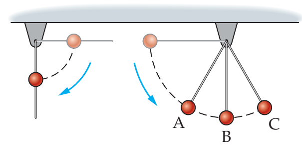
GENERAL PROBLEMS

85. • **CE** A uniform disk stands upright on its edge, and rests on a sheet of paper placed on a tabletop. If the paper is pulled horizontally to the right, as in **Figure 11–36**, **(a)** does the disk rotate clockwise or counterclockwise about its center? Explain. **(b)** Does the center of the disk move to the right, move to the left, or stay in the same location? Explain.



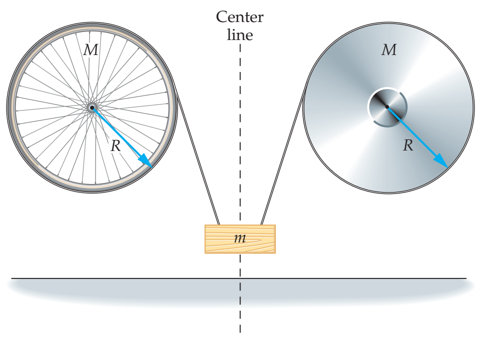
▲ **FIGURE 11–36** Problem 85

86. • **CE** Consider the two rotating systems shown in **Figure 11–37**, each consisting of a mass m attached to a rod of negligible mass pivoted at one end. On the left, the mass is attached at the midpoint of the rod; to the right, it is attached to the free end of the rod. The rods are released from rest in the horizontal position at the same time. When the rod to the left reaches the vertical position, is the rod to the right not yet vertical (location A), vertical (location B), or past vertical (location C)? Explain.



▲ **FIGURE 11–37** Problem 86

87. • **CE Predict/Explain** A disk and a hoop (bicycle wheel) of equal radius and mass each have a string wrapped around their circumferences. Hanging from the strings, halfway between the disk and the hoop, is a block of mass m, as shown in **Figure 11–38**. The disk and the hoop are free to rotate about their centers. When the block is allowed to fall, does it stay on the center line, move toward the right, or move toward the left? **(b)** Choose the *best explanation* from among the following:
   I. The disk is harder to rotate, and hence its angular acceleration is less than that of the wheel.
   II. The wheel has the greater moment of inertia and unwinds more slowly than the disk.
   III. The system is symmetric, with equal mass and radius on either side.



▲ **FIGURE 11–38** Problem 87

88. • **CE** A beetle sits at the rim of a turntable that is at rest but is free to rotate about a vertical axis. Suppose the beetle now begins to walk around the perimeter of the turntable. Does the beetle move forward, backward, or does it remain in the same location relative to the ground? Answer for two different cases, **(a)** the turntable is much more massive than the beetle and **(b)** the turntable is massless.

89. • **CE** A beetle sits near the rim of a turntable that is rotating without friction about a vertical axis. The beetle now begins to walk toward the center of the turntable. As a result, does the angular speed of the turntable increase, decrease, or stay the same? Explain.

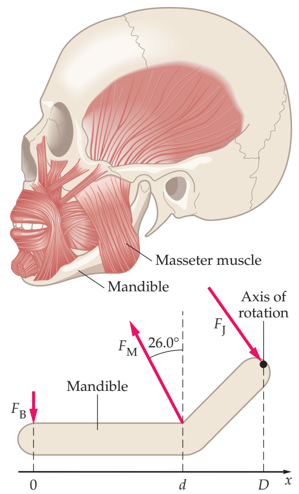
90. • **CE** Suppose the Earth were to magically expand, doubling its radius while keeping its mass the same. Would the length of the day increase, decrease, or stay the same? Explain.

91. • After getting a drink of water, a hamster jumps onto an exercise wheel for a run. A few seconds later the hamster is running in place with a speed of 1.3 m/s. Find the work done by the hamster to get the exercise wheel moving, assuming it is a hoop of radius 0.13 m and mass 6.5 g.

92. •• A 47.0-kg uniform rod 4.25 m long is attached to a wall with a hinge at one end. The rod is held in a horizontal position by a wire attached to its other end. The wire makes an angle of 30.0° with the horizontal, and is bolted to the wall directly above the hinge. If the wire can withstand a maximum tension of 1450 N before breaking, how far from the wall can a 68.0-kg person sit without breaking the wire?

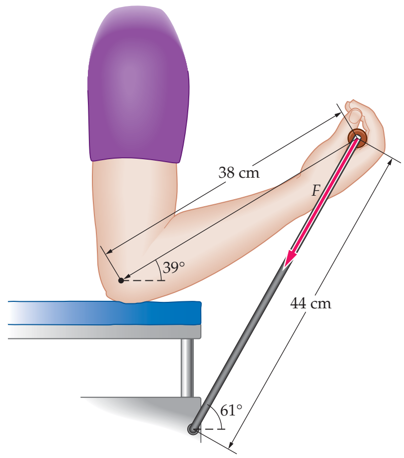
93. •• **IP** A puck attached to a string moves in a circular path on a frictionless surface, as shown in Figure 11–34. Initially, the speed of the puck is v and the radius of the circle is r. If the string passes through a hole in the surface, and is pulled downward until the radius of the circular path is $r/2$, **(a)** does the speed of the puck increase, decrease, or stay the same? **(b)** Calculate the final speed of the puck.

94. •• **BIO The Masseter Muscle** The masseter muscle, the principal muscle for chewing, is one of the strongest muscles for its size in the human body. It originates on the lower edge of the zygomatic arch (cheekbone) and inserts in the angle of the mandible. Referring to the lower diagram in **Figure 11–39**, where $d = 7.60$ cm and $D = 10.85$ cm, **(a)** find the torque produced about the axis of rotation by the masseter muscle. The force exerted by the masseter muscle is $F_M = 455$ N. **(b)** Find the biting force, F_B, exerted on the mandible by the upper teeth. Find **(c)** the horizontal and **(d)** the vertical component of the force F_J exerted on the mandible at the joint where it attaches to the skull. Assume that the mandible is in static equilibrium, and that upward is the positive vertical direction.



▲ **FIGURE 11–39** Problem 94

95. •• **Exercising the Biceps** You are designing exercise equipment to operate as shown in **Figure 11–40**, where a person pulls upward on an elastic cord. The cord behaves like an ideal spring and has an unstretched length of 31 cm. If you would like the torque about the elbow joint to be 81 N·m in the position shown, what force constant, k, is required for the cord?



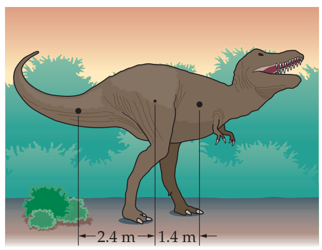
▲ **FIGURE 11–40** Problem 95

96. •• **Horsepower of a Car** Auto mechanics use the following formula to calculate the horsepower (HP) of a car engine:

$$\text{HP} = \text{Torque} \cdot \text{RPM}/C$$

In this expression, Torque is the torque produced by the engine in ft·lb, RPM is the angular speed of the engine in revolutions per minute, and C is a dimensionless constant. **(a)** Find the numerical value of C. **(b)** The Shelby Series 1 engine is advertised to generate 320 hp at 6500 rpm. What is the corresponding torque produced by this engine? Give your answer in ft·lb.

97. •• **Balancing a *T. rex*** Paleontologists believe that *Tyrannosaurus rex* stood and walked with its spine almost horizontal, as indicated in **Figure 11–41**, and that its tail was held off the ground to balance its upper torso about the hip joint. Given that the total mass of *T. rex* was 5400 kg, and that the placement of the center of mass of the tail and the upper torso was as shown in Figure 11–41, find the mass of the tail required for balance.

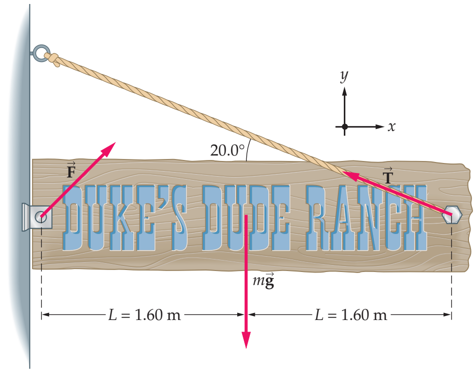


▲ **FIGURE 11–41** Problem 97

98. •• **IP** You hold a uniform, 28-g pen horizontal with your thumb pushing down on one end and your index finger pushing upward 3.5 cm from your thumb. The pen is 14 cm long. **(a)** Which of these two forces is greater in magnitude? **(b)** Find the two forces.

99. •• In Active Example 11–3, suppose the ladder is uniform, 4.0 m long, and weighs 60.0 N. Find the forces exerted on the ladder when the person is **(a)** halfway up the ladder and **(b)** three-fourths of the way up the ladder.

100. •• When you arrive at Duke's Dude Ranch, you are greeted by the large wooden sign shown in **Figure 11–42**. The left end of the sign is held in place by a bolt, the right end is tied to a rope that makes an angle of 20.0° with the horizontal. If the sign is uniform, 3.20 m long, and has a mass of 16.0 kg, what are **(a)** the tension in the rope, and **(b)** the horizontal and vertical components of the force, $\vec{F}$, exerted by the bolt?

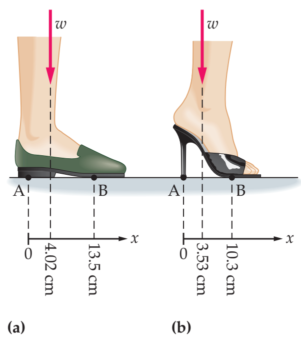


▲ **FIGURE 11–42** Problem 100

101. •• A 67.0-kg person stands on a lightweight diving board supported by two pillars, one at the end of the board, the other 1.10 m away. The pillar at the end of the board exerts a downward force of 828 N. **(a)** How far from that pillar is the person standing? **(b)** Find the force exerted by the second pillar.

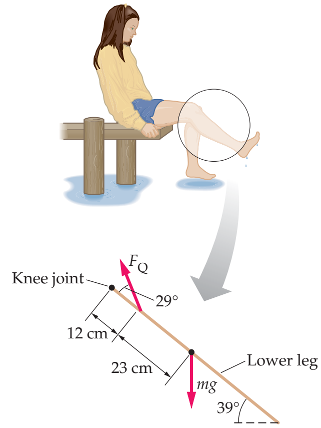
102. •• In Example 11–4, find $\vec{F}_1$ and $\vec{F}_2$ as a function of the distance, x, of the swimmer from the left end of the diving board. Assume that the diving board is uniform and has a mass of 85.0 kg.

103. •• **Flats Versus Heels** A woman might wear a pair of flat shoes to work during the day, as in **Figure 11–43 (a)**, but a pair of high heels, **Figure 11–43 (b)**, when going out for the evening. Assume that each foot supports half her weight, $w = W/2 = 279$ N, and that the forces exerted by the floor on her feet occur at the points A and B in both figures. Find the forces F_A (point A) and F_B (point B) for **(a)** flat shoes and **(b)** high heels. **(c)** How have the high heels changed the weight distribution between the woman's heels and toes?



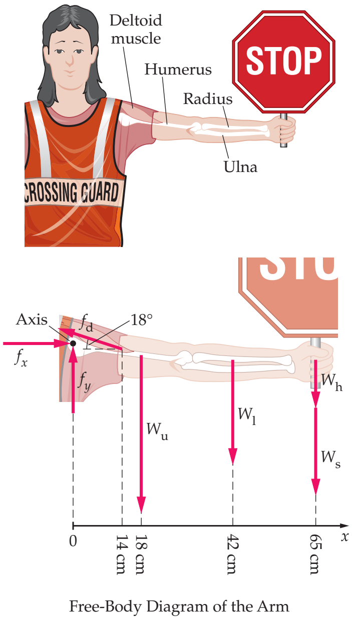
▲ **FIGURE 11–43** Problem 103

104. •• **BIO** A young girl sits at the edge of a dock by the bay, dipping her feet in the water. At the instant shown in **Figure 11–44**, she holds her lower leg stationary with her quadriceps muscle at an angle of 39° with respect to the horizontal. Use the information given in the figure, plus the fact that her lower leg has a mass of 3.4 kg, to determine the magnitude of the force, F_Q, exerted on the lower leg by the quadriceps.



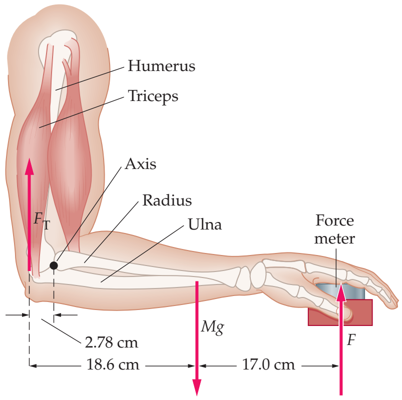
▲ **FIGURE 11–44** Problem 104

105. •• **BIO Deltoid Muscle** A crossing guard holds a STOP sign at arm's length, as shown in **Figure 11–45**. Her arm is horizontal, and we assume that the deltoid muscle is the only muscle supporting her arm. The weight of her upper arm is $W_u = 18$ N, the weight of her lower arm is $W_l = 11$ N, the weight of her hand is $W_h = 4.0$ N, and the weight of the sign is $W_s = 8.9$ N. The location where each of these forces acts on the arm is indicated in the figure. A force of magnitude f_d is exerted on the humerus by the deltoid, and the shoulder joint exerts a force on the humerus with horizontal and vertical components given by f_x, and f_y, respectively. **(a)** Is the magnitude of f_d greater than, less than, or equal to the magnitude of f_x? Explain. Find **(b)** f_d, **(c)** f_x, and **(d)** f_y. (The weights in Figure 11–45 are drawn to scale; the unknown forces are to be determined. If a force is found to be negative, its direction is opposite to that shown.)



▲ **FIGURE 11–45** Problem 105

106. •• **BIO Triceps** To determine the force a person's triceps muscle can exert, a doctor uses the procedure shown in **Figure 11–46**, where the patient pushes down with the palm of his hand on a force meter. Given that the weight of the lower arm is $Mg = 15.6$ N, and that the force meter reads $F = 89.0$ N, what is the force F_T exerted vertically upward by the triceps?

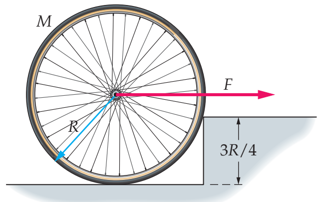


▲ **FIGURE 11–46** Problem 106

107. •• **IP** Suppose a fourth book, the same as the other three, is added to the stack of books shown in Figure 11–32. **(a)** What is the maximum overhang distance, d, in this case? **(b)** If the mass of each book is increased by the same amount, does your answer to part (a) increase, decrease, or stay the same? Explain.

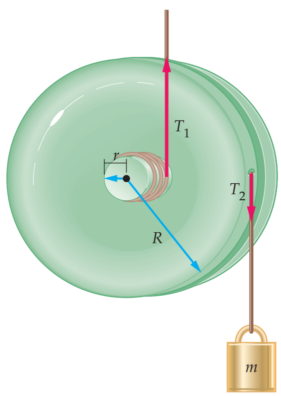
108. •• **IP** Suppose partial melting of the polar ice caps increases the moment of inertia of the Earth from $0.331\,M_E R_E^2$ to $0.332\,M_E R_E^2$. **(a)** Would the length of a day (the time required for the Earth to complete one revolution about its axis) increase or decrease? Explain. **(b)** Calculate the change in the length of a day. Give your answer in seconds.

109. ••• A bicycle wheel of radius R and mass M is at rest against a step of height $3R/4$, as illustrated in **Figure 11–47**. Find the minimum horizontal force F that must be applied to the axle to make the wheel start to rise up over the step.



▲ **FIGURE 11–47** Problem 109

110. ••• A 0.101-kg yo-yo has an outer radius R that is 5.60 times greater than the radius r of its axle. The yo-yo is in equilibrium if a mass m is suspended from its outer edge, as shown in **Figure 11–48**. Find the tension in the two strings, T_1 and T_2, and the mass m.



▲ **FIGURE 11–48** Problem 110

111. ••• In Problem 36, assume that the rod has a mass of M and that its bottom end simply rests on the floor, held in place by static friction. If the coefficient of static friction is μ_s, find the maximum force F that can be applied to the rod at its midpoint before it slips.

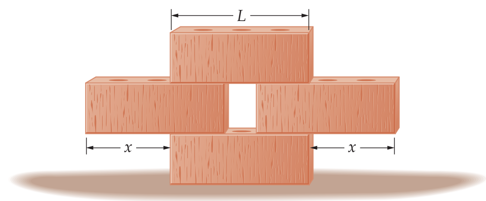
112. ••• In the previous problem, suppose the rod has a mass of 2.3 kg and the coefficient of static friction is 1/7. **(a)** Find the greatest force F that can be applied at the midpoint of the rod without causing it to slip. **(b)** Show that if F is applied 1/8 of the way down from the top of the rod, it will never slip at all, no matter how large the force F.

113. ••• A cylinder of mass m and radius r has a string wrapped around its circumference. The upper end of the string is held fixed, and the cylinder is allowed to fall. Show that its linear acceleration is $(2/3)g$.

114. ••• Repeat the previous problem, replacing the cylinder with a solid sphere. Show that its linear acceleration is $(5/7)g$.

115. ••• A mass M is attached to a rope that passes over a disk-shaped pulley of mass m and radius r. The mass hangs to the left side of the pulley. On the right side of the pulley, the rope is pulled downward with a force F. Find **(a)** the acceleration of the mass, **(b)** the tension in the rope on the left side of the pulley, and **(c)** the tension in the rope on the right side of the pulley. **(d)** Check your results in the limits $m \to 0$ and $m \to \infty$.

116. ••• **Bricks in Equilibrium** Consider a system of four uniform bricks of length L stacked as shown in **Figure 11–49**. What is the maximum distance, x, that the middle bricks can be displaced outward before they begin to tip?



▲ **FIGURE 11–49** Problem 116

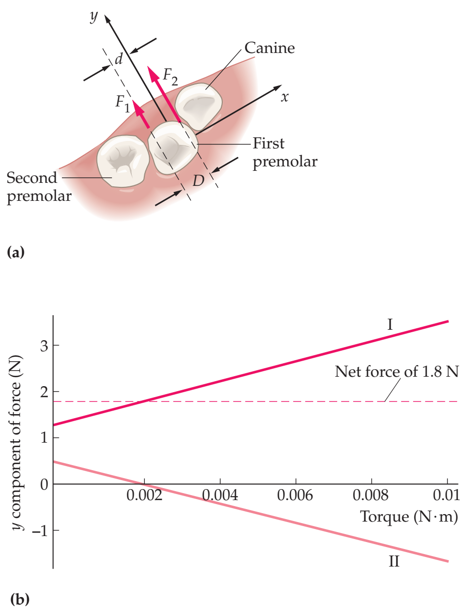
PASSAGE PROBLEMS

BIO Correcting Torsiversion

Torsiversion is a medical condition in which a tooth is rotated away from its normal position about the long axis of the root. Studies show that about 2 percent of the population suffer from this condition to some degree. For those who do, the improper alignment of the tooth can lead to tooth-to-tooth collisions during eating, as well as other problems. Typical patients display a rotation ranging from 20° to 60°, with an average around 30°.

An example is shown in **Figure 11–50 (a)**, where the first premolar is not only displaced slightly from its proper location in the negative y direction, but also rotated clockwise from its normal orientation. To correct this condition, an orthodontist might use an archwire and a bracket to apply both a force and a torque to the tooth. In the simplest case, two forces are applied to the tooth in different locations, as indicated by F_1 and F_2 in Figure 11–50 (a). These two forces, if chosen properly, can reposition the tooth by exerting a net force in the positive y direction, and also reorient it by applying a torque in the counterclockwise direction.

In a typical case, it may be desired to have a net force in the positive y direction of 1.8 N. In addition, the distances in Figure 11–50 (a) can be taken to be $d = 3.2$ mm and $D = 4.5$ mm. Given these conditions, a range of torques is possible for various values of the y components of the forces, F_{1y} and F_{2y}. For example, **Figure 11–50 (b)** shows the values of F_{1y} and F_{2y} necessary to produce a given torque, where the torque is measured about the center of the tooth (which is also the origin of the coordinate system). Notice that the two forces always add to 1.8 N in the positive y direction, though one of the forces changes sign as the torque is increased.



▲ **FIGURE 11–50** Problems 117, 118, 119, and 120

117. • The two, solid straight lines in Figure 11–50 (b) represent the two forces applied to the tooth. Which line corresponds to which force?

A. I $= F_{1y}$, II $= F_{2y}$ **B.** I $= F_{2y}$, II $= F_{1y}$

118. • What is the value of the torque that corresponds to one of the forces being equal to zero?

A. 0.0023 N·m **B.** 0.0058 N·m

C. 0.0081 N·m **D.** 0.017 N·m

119. •• Find the values of F_{1y} and F_{2y} required to give zero net torque.

A. $F_{1y} = -1.2$ N, $F_{2y} = 3.0$ N **B.** $F_{1y} = 1.1$ N, $F_{2y} = 0.75$ N

C. $F_{1y} = -0.73$ N, $F_{2y} = 2.5$ N **D.** $F_{1y} = 0.52$ N, $F_{2y} = 1.3$ N

120. •• Find the values of F_{1y} and F_{2y} required to give a net torque of 0.0099 N·m. This is a torque that would be effective at rotating the tooth.

A. $F_{1y} = -1.7$ N, $F_{2y} = 3.5$ N **B.** $F_{1y} = -3.8$ N, $F_{2y} = 5.6$ N

C. $F_{1y} = -0.23$ N, $F_{2y} = 2.0$ N **D.** $F_{1y} = 4.0$ N, $F_{2y} = -2.2$ N

INTERACTIVE PROBLEMS

121. •• **Referring to Example 11–7** Suppose the mass of the pulley is doubled, to 0.160 kg, and that everything else in the system remains the same. **(a)** Do you expect the value of T_2 to increase, decrease, or stay the same? Explain. **(b)** Calculate the value of T_2 for this case.

122. •• **Referring to Example 11–7** Suppose the mass of the cart is doubled, to 0.62 kg, and that everything else in the system remains the same. **(a)** Do you expect the value of T_2 to increase, decrease, or stay the same? Explain. **(b)** Calculate the value of T_2 for this case.

123. •• **Referring to Active Example 11–5** Suppose the child runs with a different initial speed, but that everything else in the system remains the same. What initial speed does the child have if the angular speed of the system after the collision is 0.425 rad/s?

124. •• **Referring to Active Example 11–5** Suppose everything in the system is as described in Active Example 11–5 except that the child approaches the merry-go-round in a direction that is not tangential. Find the angle θ between the direction of motion and the outward radial direction (as in Example 11–8) that is required if the final angular speed of the system is to be 0.272 rad/s.

PHYSICS IN

PERSPECTIVE

Momentum: A Conserved Quantity

When objects interact, momentum may be conserved while mechanical energy is dissipated. Why? These pages explore momentum conservation and point out key differences between momentum and mechanical energy.

1 How do linear and angular momentum relate?

The equations of linear and angular momentum are analogous, and all the principles presented on these pages apply to angular as well as linear momentum.

	Definition	Newton's 2nd law
Linear momentum:	$\vec{p} = m\vec{v}$	$\Sigma\vec{F} = m\vec{a} = \dfrac{\Delta\vec{p}}{\Delta t}$
Angular momentum:	$\vec{L} = I\vec{\omega}$	$\Sigma\vec{\tau} = I\vec{\alpha} = \dfrac{\Delta\vec{L}}{\Delta t}$

Analogous quantities

Linear	Angular
Acceleration $\vec{a}$	Angular acceleration $\vec{\alpha}$
Force $\vec{F}$	Torque $\vec{\tau}$
Velocity $\vec{v}$	Angular velocity $\vec{\omega}$
Mass m	Moment of inertia I

2 Why is momentum conserved?

Momentum conservation follows from Newton's laws.
Recall that the general form of Newton's second law relates force to momentum:

An object's change in momentum … … equals the net force acting on the object …

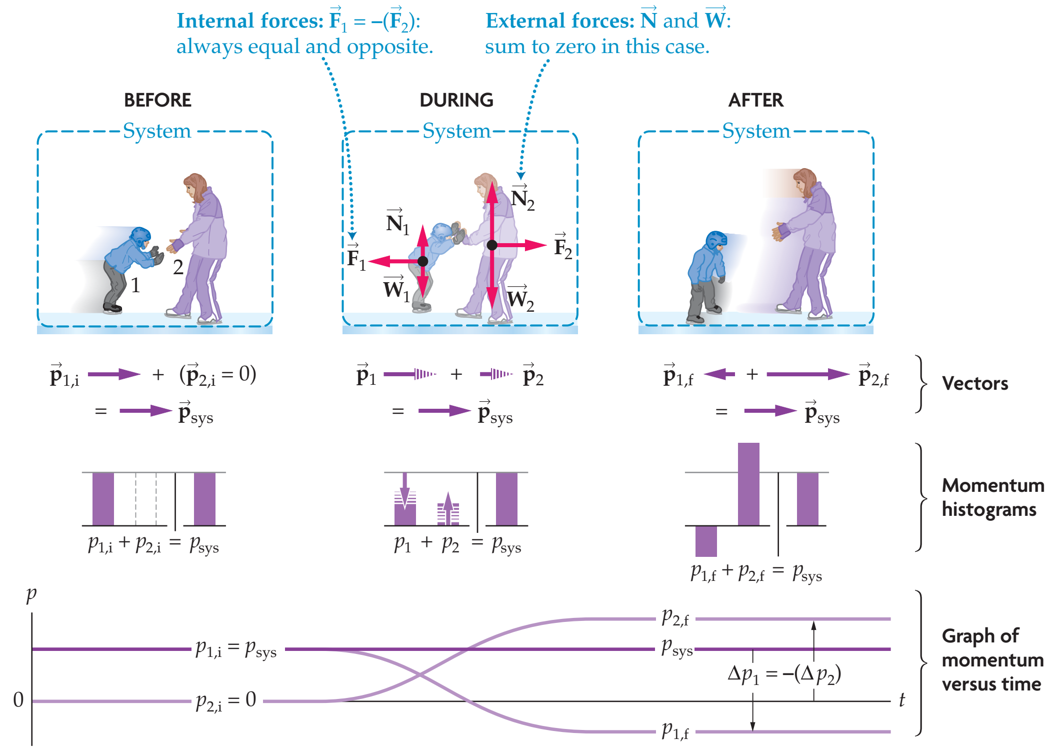
$$\Sigma\vec{F} = \Delta\vec{p}/\Delta t \quad \text{or} \quad \Delta\vec{p} = (\Sigma\vec{F})\Delta t$$

… multiplied by the time over which the force acts.

For an individual object, momentum is conserved (does not change) when the net force acting on the object is zero (that is, $\Delta\vec{p} = 0$ when $\Sigma\vec{F} = 0$).

For a system of objects, momentum conservation follows from Newton's third law:
- The momentum of a system of objects is the vector sum of the momenta of the individual objects.
- The forces between objects in the system (**internal forces**) cannot change the system's momentum because, by Newton's third law, the objects exert *equal but opposite forces* on each other, which cause *equal and opposite momentum changes.*
- **Thus, only external forces can change the momentum of a system.**

In the following interaction, the two skaters undergo equal and opposite momentum changes, whereas the system's momentum $\vec{p}_{sys}$ is conserved.



3 How can momentum be conserved when mechanical energy is not?

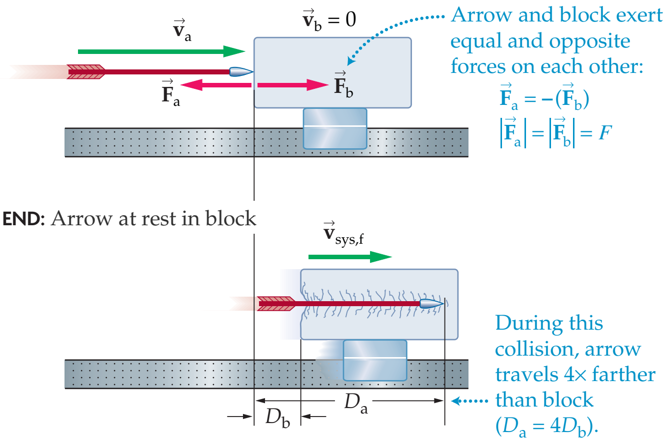
Force times time versus force times distance: Momentum change is due to *a force acting over a time* Δt, whereas changes in mechanical energy result from *a force acting over a distance D* (i.e., from work):

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}(\Delta t) \qquad \Delta E = W = F(D)$$

How do these relationships apply to the inelastic collision shown below?

Arrow shot into styrofoam block attached to air-track cart

START: Collision begins



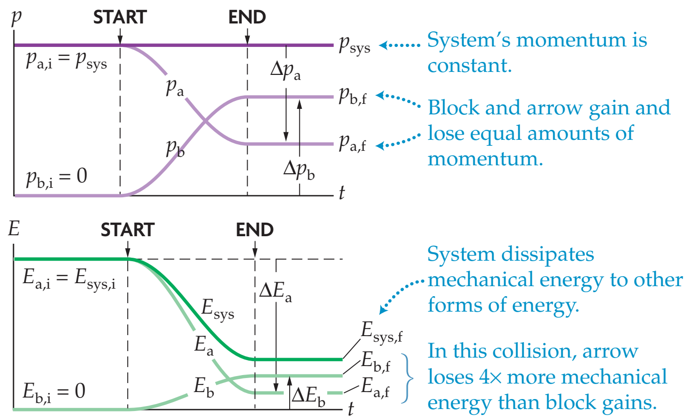
Momentum: The collision lasts the same time Δt for the arrow and block, so their mometum changes are equal and opposite:

$$\Delta\vec{\mathbf{p}}_a = \vec{\mathbf{F}}_a\Delta t = -(\vec{\mathbf{F}}_b)\Delta t = -\Delta\vec{\mathbf{p}}_b$$

Mechanical energy: The objects exert the same force magnitude F on each other, but the arrow travels farther during the collision because it penetrates the block: $D_a > D_b$. Thus, the arrow loses more mechanical energy than the block gains:

$$\Delta E_a = F_a(D_a) = -40 \text{ J} \qquad \Delta E_b = F_b(D_b) = +10 \text{ J}$$

Conclusion: The collision dissipates mechanical energy while conserving momentum, as the following graphs show:

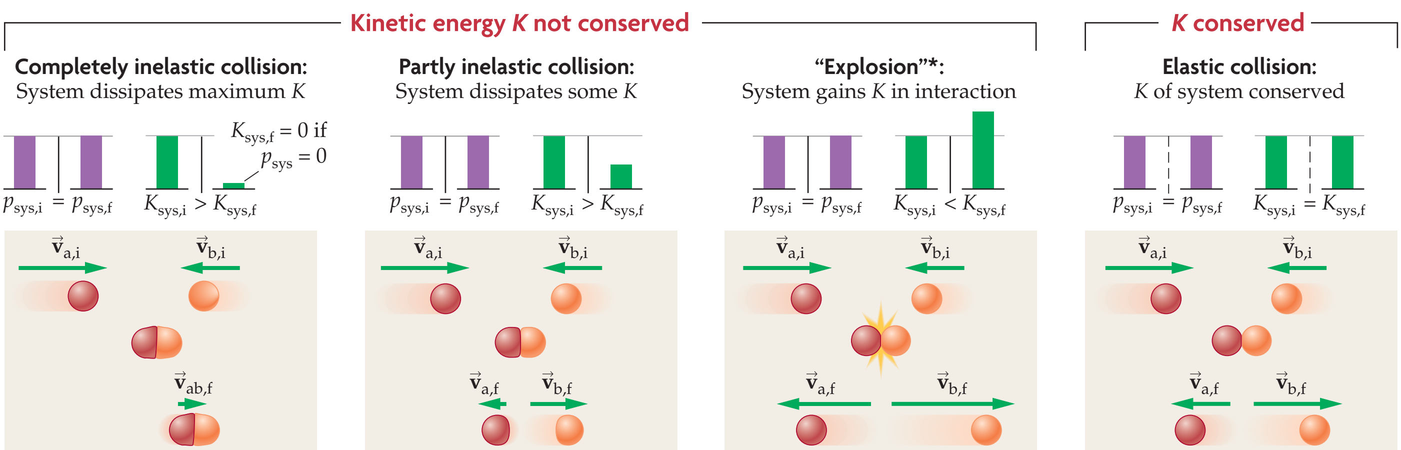


4 How does momentum conservation help us solve problems?

- You can use momentum conservation to analyze any interaction between objects for which the net external force acting on the system during the collision is zero (or is negligible compared to the internal forces).
- If the net external force is not negligible, you cannot use momentum conservation! This applies to the players at right, who push on the ground while colliding.
- For elastic collisions, you must use conservation of mechanical energy as well as conservation of momentum. (Section 9.6 solves these simultaneous equations for special cases.)

Collisions in which momentum is conserved can be categorized as follows, according to how kinetic energy changes or is conserved.

The external reaction forces of the earth on these players' feet cannot be ignored.



*Physicists use the term "explosion" to mean any interaction that adds kinetic energy to the system. Thus, the collision in Step 2 on the facing page is an explosion.